The Boundary Element Method

an introduction and application to a three-dimensional acoustical problem

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1 Introduction

1.1 Outline

In this paper we introduce the concept of boundary integral equation formulations for suitable partial differential equations and a numerical solution approach, the boundary element method. We are primarily concerned with the wave equation for single frequencies, the Helmholtz equation. The Helmholtz equation governs single frequency sound fields inside a body or outside of an arbitrary number of bodies positioned in free space. We derive the Helmholtz equation from the Wave equation and introduce necessary conditions for the uniqueness of its solution in the exterior case. Then we develope the boundary integral formulation for the Helmholtz equation and a simple numerical solution routine, the collocation method. We attend to the non-uniqueness problem of the direct boundary integral formulation with an augmented equation, the Burton-Miller formulation. The direct and the Burton-Miller formulation have numerically difficult (hyper-)singular integrals. We present a nonsingular analytical reformulation of those integrals for the collocation method case. A boundary element collocation solver for the Helmholtz equation with graphic user interface has been implemented in the framework of this paper. Observations made with this solver segue to open problems with the Burton-Miller formulation and further reading.

1.2 Motivation

The problem of incident acoustic waves being scattered by objects or the acoustic radiation from objects into the acoustic medium or a combination of both is an important problem in physics and engineering. Real world problems are for example the design of acoustic barriers for highways, the acoustic design of the interior of cars or the engeneering of loudspeakers. In contrast to the well-known finite difference methods or finite element methods, it is necessary to discretize only the boundary of the domain of interest for the boundary element method. The boundary element method therefore effectively has the advantage of reducing the dimensionality of the promblem by one. This advantage is further exemplified in the exterior scattering and radiation case. When simulating a loudspeaker in an anechoic environment via the BEM for example, the governing equation for the sound field in the infinite environment has to be solved only for the finite surface of the loudspeaker itself. This results in a highly reduced number of degrees of freedom for the numerical method.

2 Preliminaries

Throughout this text the domain is denoted U and is three-dimensional. U is always assumed to be simply connected and closed. Vectors are bold, for example $\mathbf{r} \in U$, except in directional derivatives, e.g. $\frac{\partial}{\partial r}$. The vector $\mathbf{p} \in U$ will be referred to as the observation point. The vectors $\mathbf{n}_p$ and $\mathbf{n}_q$ are the unit normals to the boundary at $\mathbf{p}$, $\mathbf{q} \in \partial U$ respectively. The normals point from ∂U into U. Note that in some references, the normals are assumed to point into the opposite direction.

2.1 Laplace operator

Throughout this text the expression ∇^2 signifies the Laplace operator, which in three-dimensional cartesian coordinates is:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2.2 Dirac delta function

The Dirac delta function or δ-function can be considered a generalized function on the real line, which is equal to zero for all real numbers exept at zero and whose integral over its domain is one.

$$\delta(x) := \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \qquad x \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

An important property of the delta function is the so called sampling property.

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, \delta(a-x)\, \mathrm{d}x = f(a), \quad \forall a \in \mathbb{R}$$

[Hos09, page 34]

The δ-distribution on $\mathbb{R}^n$:

$$\delta(x-a) := \delta(x_1 - a_1) \cdot \delta(x_2 - a_2) \cdot_{\dots} \cdot \delta(x_n - a_n), \quad x, a \in \mathbb{R}^n$$

The sampling property holds also for the n-dimensional δ-distribution.

2.3 Green's second identity

It holds for twice continuously differentiable scalar functions ψ and ϕ on U.

$$\int\limits_{U} \left(\phi(\mathbf{q})\nabla^2\psi(\mathbf{q}) - \psi(\mathbf{q})\nabla^2\phi(\mathbf{q})\right) \mathrm{d}q = \int\limits_{\partial U} \left(\psi(\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q} - \phi(\mathbf{q})\frac{\partial\psi(\mathbf{q})}{\partial n_q}\right) \mathrm{d}q$$

Note that in the case of normals pointing into the complement of U, the signs on one side would change [Ruo13, page 109].

3 Boundary integral equation formulations

In this section the concept of boundary integral equation formulations is intruduced via the example of the homogenous Laplace equation.

$$\nabla^2 \phi(\mathbf{p}) = 0, \quad \forall \mathbf{p} \in U$$

A Green's function $G(\mathbf{p}, \mathbf{q})$, also called fundamental solution, for the Laplace equation is an equation, which satisfies:

$$\nabla^2 G(\mathbf{p}, \mathbf{q}) = -\delta(\mathbf{p} - \mathbf{q}), \quad \forall \mathbf{p}, \mathbf{q} \in U$$

It is verifiable that $G(\mathbf{p}, \mathbf{q}) = \frac{1}{4\pi} \cdot \frac{1}{\|\mathbf{p}-\mathbf{q}\|_2}$ satisfies above requirement. Note that $G(\mathbf{p}, \mathbf{q})$ is singular for $\| \mathbf{p} - \mathbf{q} \| = 0$ with a first order pole. This prevents us from inserting the Green's function into Green's second identity directly. Therefore $G(\mathbf{p}, \mathbf{q})$ is modified as follows [Bin, page 78]:

$$G_\varepsilon(\mathbf{p}, \mathbf{q}) = \frac{1}{4\pi} \cdot \frac{1}{\| \mathbf{p} - \mathbf{q} \|_2 + \varepsilon}, \quad \varepsilon > 0 \tag{3.1}$$

Inserting (3.1) into Green's second theorem leads to:

$$\int\limits_U \left(\phi(\mathbf{q}) \nabla^2 G_\varepsilon(\mathbf{p}, \mathbf{q}) - G_\varepsilon(\mathbf{p}, \mathbf{q}) \underbrace{\nabla^2 \phi(\mathbf{q})}_{=0}\right) \mathrm{d}q = \int\limits_{\partial U} \left(G_\varepsilon(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} - \phi(\mathbf{q}) \frac{\partial G_\varepsilon(\mathbf{p}, \mathbf{q})}{\partial n_q} \right) \mathrm{d}q$$

For $\lim \varepsilon \rightarrow 0$ the above turns into:

$$-\int\limits_U \phi(\mathbf{q}) \delta(\mathbf{p} - \mathbf{q})\, \mathrm{d}q = \int\limits_{\partial U} \left(G(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} - \phi(\mathbf{q}) \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \right) \mathrm{d}q$$

Finally by the virtue of the δ-distributions sampling property, we arrive at the boundary equation formulation for the homogenous Laplace equation.

$$\phi(\mathbf{p}) = \int\limits_{\partial U} \left(\phi(\mathbf{q}) \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} - G(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q$$

4 The Helmholtz equation

In this section our partial differential equation of interest is introduced via the homogenous wave equation.

4.1 The Wave equation

We assume that the acoustic medium is a homogenous and isotropic fluid. The acoustic pressure field in such a medium is governed by the linear wave equation

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0,$$

irrespective of the shape of the domain, where p is the time-dependent acoustic pressure and c the speed of wave propagation [Kir98, page 12]. The vector valued time-dependent particle velocity $\mathbf{u}$ is governed via a similar linear wave equation

$$\nabla^2 \mathbf{u} - \frac{1}{c^2}\frac{\partial^2 \mathbf{u}}{\partial t^2} = 0.$$

It is generally is not easy, when having already solved for one unknown, $\mathbf{u}$ or p, to find the other quantity. Therefore it is more convenient to solve the wave equation for an abstract scalar field velocity potential of the form [EEH69, page 168]:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = 0 \tag{4.1}$$

When the sound pressure p or the particle velocity $\mathbf{u}$ are sought, they can be directly calculated with the following two equations, that relate them to the velocity potential Ψ:

$$\mathbf{u} = \nabla \Psi$$

and

$$p = -\rho \frac{\partial}{\partial t}\Psi,$$

where ρ is the density of the medium.

4.2 Deriving the Helmholtz equation

In this text only periodic solutions of the wave equation are considered and a separability of variables in the velocity potential $\Psi(x, t)$ is assumed. Therefore the time-dependent velocity potential $\Psi(x,t)$ can be broken down into a sum of components, each of the form

$$\psi(x,t) = Re(\phi(x)e^{-i\omega t}), \quad x \in U, \tag{4.2}$$

ω being the angular frequency ($\omega = 2\pi v$, v is the frequency in hertz) and $\phi(x)$ the time-independent veocity potential [Kir98, page 12]. Expression (4.2) can be substituted into (4.1), which leads directly to the Helmholtz equation

$$\nabla^2\phi(x) + k^2\phi(x) = 0,$$

with $k^2 = \frac{\omega^2}{c^2}$. k is called the wavenumber. For simplicity only time harmonic ($e^{-i\omega t}$ time dependence) acoustic propagation and scattering is considered in the following. More complex variations in time can be modelled by combining multiple harmonic time dependences, thus reconstructing the time-dependent velocity potential via Fourier synthesis with single frequency components of the form (4.2).
The relation between the wavenumber k and the frequency of waves in a medium is determined by its wave propagation speed.

$$k = \frac{2\pi v}{c} = \frac{2\pi}{\lambda}$$

The time-independent sound pressure $p(x)$ and time-independent particle velocity $\mathbf{u}(x)$ relate to the time-independent velocity potential ϕ via

$$\mathbf{u} = \nabla\phi(x)$$

and

$$p(x) = -\rho\frac{\partial}{\partial t}\phi(x) = i\rho\omega\phi(x),$$

where ρ is the density of the medium. For reference for the two equations above, see [Juh94, page 18] and [Kir98, page 12] respectively.

4.3 Domain considerations

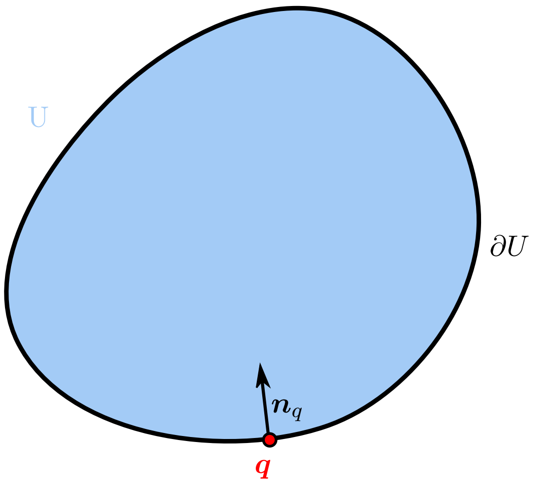

Figure 4.1: Interior acoustic problem

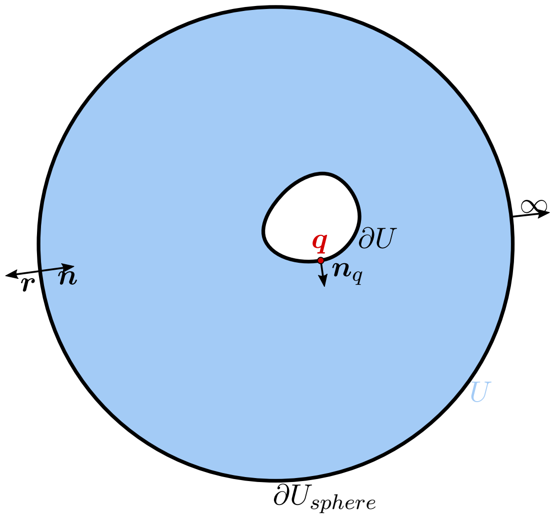

Figure 4.2: Exterior acoustic problem

We distinguish two types of acoustic problems, the interior and the exterior case. In the three-dimensional interior case the domain can be thought of as the volume enclosed inside a shell (figure 4.1). In the three-dimensional exterior case the domain can be modelled as the volume outside of closed geometries, further bounded by an infinite sphere (figure 4.2). A physical approach dictates, that no sound radiation is returned from the infinite distance (the boundary of the outer sphere ∂U_{sphere}) into the domain. This assumption is modelled by the Sommerfeld radiation condition, which can be thought of as a boundary condition at infinity.

4.4 Impedance boundary condition

By the end of this paper, we will have derived a boundary element method for the Helmholtz equation on arbitrary connected closed three-dimensional domains with general boundary conditions:

$$a(\mathbf{q})\phi(\mathbf{q}) + b(\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q} = f(\mathbf{q}), \quad \mathbf{q} \in \partial U,$$

with a, b and f being complex-valued functions, defined on the boundary. Clearly $a(\mathbf{q}) = b(\mathbf{q}) = 0$ for any $\mathbf{q} \in \partial U$ is not admissible. The most relevant boundary condition for physical modelling is the impedance boundary condition [CL07, page 4]:

$$\frac{\partial\phi(\mathbf{q})}{\partial n_q} + ik\beta\phi(\mathbf{q}) = f(\mathbf{q})$$

β is the relative *surface admittance* of the boundary at the specified frequency (or wavenumber). f is zero for pure scattering problems and nonzero, where there is a radiating boundary. For $\beta = 0$ and $f = 0$ we have *Neumann* boundary conditions:

$$\frac{\partial \phi(\mathbf{q})}{\partial n_q} = 0 \tag{4.3}$$

Surfaces with *Neumann* boundary conditions are called *sound hard* or fully reflective. As $\frac{\partial \phi(\mathbf{q})}{\partial n_q}$ can be interpreted as the surface particle velocity of the boundary at $\mathbf{q}$ [Kir98, page 73], (4.3) basically states that the particles at $\mathbf{q}$ travel into the boundary with the same velocity as they are travelling into the opposite direction.

4.5 Sommerfeld radiation condition

"The sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into ... the field."

Arnold Sommerfeld

Generally in exterior radiation problems, it is assumed that the energy is only outgoing into the distance and not being reflected. As the homogenous helmholtz equation does not imply a damping behaviour for the solution ϕ, an unphysical resonance in the domain could occur. Therefore two boundary conditions at infinity are imposed to ensure that all energy is going outward and is spread. These boundary conditions are called the Sommerfeld radiation and finiteness condition [H.S92, page 393] .

The three-dimensional finiteness condition:

$$\phi(\mathbf{q}) = O(r^{-1})$$

The three-dimensional radiation condition:

$$\lim_{r\to\infty} r \left(\frac{\partial \phi(\mathbf{q})}{\partial r} - ik\phi(\mathbf{q}) \right) = 0, \tag{4.4}$$

with $\frac{\partial}{\partial r}$ being the radial direction derivative and $r = \| \mathbf{q} \|$. Note that for $r\to\infty$, the radial derivative $\frac{\partial \phi(\mathbf{q})}{\partial r}$ equals the negative normal derivative $-\frac{\partial \phi(\mathbf{q})}{\partial n_q}$ for $\mathbf{q} \in \partial U_{sphere}$.

A physical basis for the finiteness condition is that the finite outwards travelling energy is spread over an ever larger and larger sphere surface ∂U_{sphere} of area $4\pi r^2$. So the energy decreases antiproportional to r^2. The radiation condition stems from the fact, that in the far distance the outgoing wave appears locally as a free-air plane wave travelling in the direction r [CL07, page 6].

We show now the impact of Sommerfelds conditions on a boundary integral of the type as in section (3) over the outer infinite sphere surface. It is assumed here, that the function G also satisfies above conditions. It holds:

$$\begin{aligned} &\int\limits_{\partial U_{sphere}} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \\ &= \int\limits_{\partial U_{sphere}} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} + ikG(\mathbf{p},\mathbf{q}) \right) \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \left(\frac{\partial \phi(\mathbf{q})}{\partial n_q} + ik\phi(\mathbf{q}) \right) \mathrm{d}q \end{aligned} \tag{4.5}$$

It follows, that the integrand on the right-hand side of (4.5) decreases in the order of $o(r^{-2})$. As the surface of the sphere increases proportional to r^2, the above boundary integral evaluates to zero for $r \to \infty$. This result allows us to omit the integral over the bounding sphere in exterior radiation and scattering problems and the exterior case will be handled as such for the remainder of this text.

4.6 Fundamental solutions of the Helmholtz equation

The fundamental solution or Green's function $G(\mathbf{p},\mathbf{q})$ for the Helmholtz equation is the equation, which satisfies

$$(\nabla^2 + k^2)G(\mathbf{p},\mathbf{q}) = -\delta(\mathbf{p}-\mathbf{q}), \quad \mathbf{p},\mathbf{q} \in U \tag{4.6}$$

The following are eligible functions.

$$G_k^{\pm}(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi} \frac{e^{\pm ik\|\mathbf{p}-\mathbf{q}\|_2}}{\| \mathbf{p}-\mathbf{q} \|_2} \tag{4.7}$$

[Kir98, page 32]
The so-called outgoing wave Green's function:

$$G_k(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi} \frac{e^{ik\|\mathbf{p}-\mathbf{q}\|_2}}{\| \mathbf{p}-\mathbf{q} \|_2} \tag{4.8}$$

The so-called incoming wave Green's function:

$$G_k^{-}(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi} \frac{e^{-ik\|\mathbf{p}-\mathbf{q}\|_2}}{\| \mathbf{p}-\mathbf{q} \|_2} \tag{4.9}$$

Note that (4.9) does not satisfy the Sommerfeld radiation condition. Therefore G or G_k will always denote the outgoing wave Green's function (4.8) for the remainder of this text. We will show the compliance of G with Sommerfeld's radiation condition in a following section.

We introduce the following definitions to enable us to break down the derivatives of G:

$$\mathbf{r}(\mathbf{p},\mathbf{q}) \coloneqq \mathbf{p} - \mathbf{q}$$

$$r(\mathbf{p},\mathbf{q}) \coloneqq \| \mathbf{r}(\mathbf{p},\mathbf{q}) \|_2$$

Now G with regard to $r(\mathbf{p},\mathbf{q})$ is:

$$G_k(r(\mathbf{p},\mathbf{q})) = \frac{1}{4\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})}$$

4.6.1 Normal derivatives of r

As stated earlier, the vectors $\mathbf{n}_p$ and $\mathbf{n}_q$ are the unit normals to the boundary at $\mathbf{p}$ and $\mathbf{q} \in \partial U$ respectively. The normals always point from ∂U *into* U. Note that in all instances where $\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}$ will be used, the variable of differentiation is $\mathbf{q}$. Therefore:

$$\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} = -\frac{\mathbf{r}(\mathbf{p},\mathbf{q}) \cdot \mathbf{n}_q}{r(\mathbf{p},\mathbf{q})} \tag{4.10}$$

The situation is different with $\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p}$. The term will appear later in a boundary integral formulation, which is itself a function of $\mathbf{p}$.

$$\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} = \frac{\mathbf{r}(\mathbf{p},\mathbf{q}) \cdot \mathbf{n}_p}{r(\mathbf{p},\mathbf{q})} \tag{4.11}$$

$$\frac{\partial^2 r(\mathbf{p},\mathbf{q})}{\partial n_p \partial n_q} = -\frac{1}{r(\mathbf{p},\mathbf{q})} \left(\mathbf{n}_p \cdot \mathbf{n}_q + \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} \right) \tag{4.12}$$

4.6.2 Derivatives of G(r)

$$G_k(r(\mathbf{p},\mathbf{q})) = \frac{1}{4\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})}$$

$$\frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r} = \frac{G_k(r(\mathbf{p},\mathbf{q}))}{r(\mathbf{p},\mathbf{q})}(ikr(\mathbf{p},\mathbf{q}) - 1)$$

$$\frac{\partial^2 G_k(r(\mathbf{p},\mathbf{q}))}{\partial r^2} = \frac{G_k(r(\mathbf{p},\mathbf{q}))}{r(\mathbf{p},\mathbf{q})^2}(2 - 2ikr(\mathbf{p},\mathbf{q}) - k^2 r(\mathbf{p},\mathbf{q})^2)$$

4.6.3 Normal derivatives of the Greens function

These normal derivatives appear in the boundary integral formulations and consist of components from the previous two subsections.

$$\frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial n_q} = \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}$$

$$\frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial n_p} = \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p}$$

$$\frac{\partial^2 G_k(r(\mathbf{p},\mathbf{q}))}{\partial n_p \partial n_q} = \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\frac{\partial^2 r(\mathbf{p},\mathbf{q})}{\partial n_p \partial n_q} + \frac{\partial^2 G_k(r(\mathbf{p},\mathbf{q}))}{\partial r^2}\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p}\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}$$

4.6.4 G as a solution to the Helmholtz equation

We will briefly show $\nabla^2 G(\mathbf{p},\mathbf{q}) = -k^2 G(\mathbf{p},\mathbf{q})$ for $\mathbf{p} \neq \mathbf{q}$. In the following r_x, r_y and r_z are the cartesian components of the vector $\mathbf{r}(\mathbf{p},\mathbf{q}) = \mathbf{p} - \mathbf{q}$.

$$\frac{\partial G_k(\mathbf{p},\mathbf{q})}{\partial x} = \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial x}$$

$$\frac{\partial^2 G_k(\mathbf{p},\mathbf{q})}{\partial x^2} = \frac{\partial^2 G_k(r(\mathbf{p},\mathbf{q}))}{\partial r^2}\left(\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial x}\right)^2 + \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\left(\frac{\partial^2 r(\mathbf{p},\mathbf{q})}{\partial x^2}\right)$$

$$\frac{\partial^2 r(\mathbf{p},\mathbf{q})}{\partial x^2} = \frac{r_y^2 + r_z^2}{r(\mathbf{p},\mathbf{q})^3}$$

The above is valid, regardless of whether $\mathbf{p}$ or $\mathbf{q}$ are the variable of differentiation. Therefore:

$$\begin{aligned}
\nabla^2 G_k(\mathbf{p},\mathbf{q}) =& \frac{\partial^2 G_k(r(\mathbf{p},\mathbf{q}))}{\partial r^2}\left(\nabla r(\mathbf{p},\mathbf{q}) \cdot \nabla r(\mathbf{p},\mathbf{q})\right) + \frac{\partial G_k(r(\mathbf{p},\mathbf{q}))}{\partial r}\nabla^2 r(\mathbf{p},\mathbf{q}) \\
=& \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})^3}(2 - 2ikr(\mathbf{p},\mathbf{q}) - k^2 r(\mathbf{p},\mathbf{q})^2)\frac{r_x^2 + r_y^2 + r_z^2}{r(\mathbf{p},\mathbf{q})^2} \\
&+ \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})^2}(ikr(\mathbf{p},\mathbf{q}) - 1)\frac{2r_x^2 + 2r_y^2 + 2r_z^2}{r(\mathbf{p},\mathbf{q})^3} \\
=& \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})^5}(-k^2 r(\mathbf{p},\mathbf{q})^2)(r_x^2 + r_y^2 + r_z^2) \\
=& -k^2 \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})} \\
=& -k^2 G_k(\mathbf{p},\mathbf{q})
\end{aligned}$$

What we wanted to show.

4.6.5 Verifying the radiation condition assumption

For (4.5) we assumed, that the Sommerfeld radiation condition (4.4) applies to the Greens function G. We verify that the outgoing wave Green's function $G_k(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})}$ actually satisfies the condition. For the following let $r = r(\mathbf{p},\mathbf{q})$.

$$\begin{aligned}
\lim_{r\to\infty} r\left(-\frac{\partial G(r)}{\partial n_q} - ikG(r)\right) &= \lim_{r\to\infty} r\left(-\frac{\partial G(r)}{\partial r}\underbrace{\frac{\partial r}{\partial n_q}}_{=-1} - ikG(r)\right) \\
&= \lim_{r\to\infty} r\left(\frac{e^{ikr}}{4\pi r^2}(ikr-1) - ik\frac{e^{ikr}}{4\pi r}\right) \\
&= \lim_{r\to\infty}\left(ik\frac{e^{ikr}}{4\pi} - \frac{e^{ikr}}{4\pi r} - ik\frac{e^{ikr}}{4\pi}\right) \\
&= \lim_{r\to\infty} -\frac{e^{ikr}}{4\pi r} \\
&= 0
\end{aligned}$$

The above result confirms, that an integral of type (4.5) with $G = \frac{1}{4\pi}\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})}$ over ∂U_{sphere} can be omitted for the remainder of the text.

5 Boundary integral formulation of the Helmholtz equation

In this chapter we derive the boundary integral equation formulation for the Helmholtz equation. We'll get to the ∂U-integral by applying Green's second identity in a nonrigorous way and then partially evaluating the boundary integral. Like in the Laplace case, we could also treat the singular behaviour of G, but the following scheme will serve us well for examining additional acoustic sources and aspects of the numerical implementation later on.

In the following it is assumed, that the boundary ∂U is piecewise smooth. The observation point $\mathbf{p}$ is assumed to be in the interior U° of the domain U. Again, the vector $\mathbf{n}_q$ is the unit normal to the boundary at $\mathbf{q} \in \partial U$, directed into U.

It holds that:

$$\int_U G(\mathbf{p},\mathbf{q})\underbrace{(\nabla^2 + k^2)\,\phi(\mathbf{q})}_{=0} - (\nabla^2 + k^2)G(\mathbf{p},\mathbf{q})\,\phi(\mathbf{q})\,\mathrm{d}q = \int_U G(\mathbf{p},\mathbf{q})\,\nabla^2\phi(\mathbf{q})\,\mathrm{d}q - \int_U \nabla^2 G(\mathbf{p},\mathbf{q})\,\phi(\mathbf{q})\,\mathrm{d}q$$

By applying Green's second identity nonrigorously to the right hand side, it follows:

$$-\int_U (\nabla^2 + k^2)G(\mathbf{p},\mathbf{q})\,\phi(\mathbf{q})\,\mathrm{d}q \simeq \int_{\partial U} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial \phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q$$

$$\int_U \delta(\mathbf{p}-\mathbf{q})\,\phi(\mathbf{q})\,\mathrm{d}q \simeq \int_{\partial U} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial \phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q$$

By virtue of the sampling property of the δ-distribution, the boundary integral equation for the observation point $\mathbf{p}$ in the interior U° of the domain U follows.

$$\phi(\mathbf{p}) \simeq \int_{\partial U} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial \phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q$$

Since Green's second identity is only valid for two continuously differentiable functions, it is not directly applicable to $\phi(\mathbf{q})$ and $G(\mathbf{p},\mathbf{q})$ on U, because $G(\mathbf{p},\mathbf{q})$ and $\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}$ are singular at $||\mathbf{p}-\mathbf{q}||_2 = 0$ with a first and second order pole respectively. Therefore we modify the domain and boundary at $\mathbf{p}$ by cutting out a ball $B_\varepsilon^{\mathbf{p}}$ around $\mathbf{p}$ with radius ε. Green's second identity is valid for the modified domain U_ε as the singularity is avoided. The approach is then to take the radius ε to zero and evaluate the integral of ∂U_ε in two parts. First the part that bounds the intersection of $B_\varepsilon^{\mathbf{p}}$ and U and then the rest of ∂U_ε. This strategy is derived from [CL07, page 13].

The definition of the ball:

$$B_\varepsilon^{\mathbf{p}} := \{\mathbf{q} \in U \mid \ \| \mathbf{q} - \mathbf{p} \| < \varepsilon \ \}$$

The modified domain is now:

$$U_\varepsilon = U \setminus B_\varepsilon^{\mathbf{p}} = U \setminus \{\mathbf{q} \in U \mid \ \| \mathbf{q} - \mathbf{p} \| < \varepsilon \ \}$$

Assumed that $\phi \in C^2$ on U, $\phi(\mathbf{q})$ and $G(\mathbf{p}, \mathbf{q})$ satisfy Greens second theorem on the modified Boundary ∂U_ε for $\mathbf{q} \in \partial U_\varepsilon$ and $\mathbf{p} \in U \setminus U_\varepsilon$. It follows therefore:

$$\begin{aligned} \int\limits_{\partial U_\varepsilon} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q &= \int\limits_{U_\varepsilon} \left(G(\mathbf{p},\mathbf{q}) \, \nabla^2 \phi(\mathbf{q}) - \nabla^2 G(\mathbf{p},\mathbf{q}) \, \phi(\mathbf{q}) \right) \mathrm{d}q \\ &= \int\limits_{U_\varepsilon} G(\mathbf{p},\mathbf{q}) \underbrace{(\nabla^2 + k^2) \, \phi(\mathbf{q})}_{=0} \, \mathrm{d}q \\ &\quad - \int\limits_{U_\varepsilon} \underbrace{(\nabla^2 + k^2) \, G(\mathbf{p},\mathbf{q})}_{=0} \, \phi(\mathbf{q}) \, \mathrm{d}q \\ &= 0 \end{aligned} \tag{5.1}$$

In the following subsections the two terms of the left-hand side of (5.1) will be examined separately for $\mathbf{p} \in U^{\mathrm{o}}$ and $\mathbf{p} \in \partial U$.

5.1 Observation point inside the domain

First we consider the case of the observation point in the interior of the domain. For $\lim \varepsilon \to 0$ $\partial B_\varepsilon^{\mathbf{p}}$ doesn't intersect ∂U. Therefore ∂U_ε can be decomposed as follows:

$$\partial U_\varepsilon = \partial U \cup \partial B_\varepsilon^{\mathbf{p}} \quad \text{with} \quad \partial B_\varepsilon^{\mathbf{p}} := \{\mathbf{q} \in U \mid \ \| \mathbf{q} - \mathbf{p} \| = \varepsilon \ \}$$

We now evaluate the terms of left-hand side of (5.1) for ∂U and $\partial B_\varepsilon^{\mathbf{p}}$ separately. The boundary of the sphere $\partial B_\varepsilon^{\mathbf{p}}$ will be represented through spherical coordinates. Let $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$ be pairwise orthogonal unit vectors at $\mathbf{p}$. Note that the vectors form a local orthonomal base at $\mathbf{p}$, see figure 5.1. This coordinate transform strategy is a slight variaton of [MZHT10, page 197].

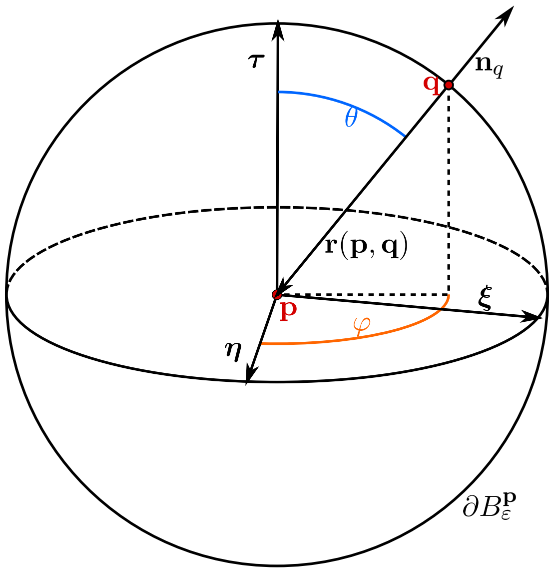

Figure 5.1: The ball $B_\varepsilon^{\mathbf{p}}$

A point $\mathbf{q} \in \partial B_\varepsilon^{\mathbf{p}}$ in the spherical coordinates of the orthonormal base of $B_\varepsilon^{\mathbf{p}}$:

$$\mathbf{q} = \varepsilon(\sin(\theta)\cos(\varphi)\boldsymbol{\eta} + \sin(\theta)\sin(\varphi)\boldsymbol{\xi} + \cos(\theta)\boldsymbol{\tau})$$

Note that the Jacobian determinant of the transformation from the original cartesian coordinates to the spherical coordinates of the local orthonormal base is $\varepsilon^2 \sin(\theta)$.
The separate terms of equation (5.1):

$$H_1 \coloneqq \int\limits_{\partial B_\varepsilon^{\mathbf{p}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) \,\mathrm{d}q$$

$$H_2 \coloneqq -\int\limits_{\partial B_\varepsilon^{\mathbf{p}}} G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \,\mathrm{d}q$$

$$I_1 \coloneqq \int\limits_{\partial U} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) \,\mathrm{d}q$$

$$I_2 \coloneqq -\int\limits_{\partial U} G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \,\mathrm{d}q$$

$$
\begin{aligned}
H_1 &= \int\limits_{\partial B_\varepsilon^{\mathbf{p}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q})\,\mathrm{d}q \\
&= \int\limits_0^{2\pi}\int\limits_0^{\pi} \left(\frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}}_{=1} \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int\limits_0^{2\pi}\int\limits_0^{\pi} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon^2}(ik\varepsilon - 1)\phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\frac{e^{ik\varepsilon}}{4\pi}(1 - ik\varepsilon) \int\limits_0^{2\pi}\int\limits_0^{\pi} (\phi(\mathbf{q}) - \phi(\mathbf{p}))\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad - \frac{e^{ik\varepsilon}}{4\pi}(1 - ik\varepsilon) \int\limits_0^{2\pi}\int\limits_0^{\pi} \phi(\mathbf{p})\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\frac{e^{ik\varepsilon}}{4\pi}(1 - ik\varepsilon) \int\limits_0^{2\pi}\int\limits_0^{\pi} (\phi(\mathbf{q}) - \phi(\mathbf{p}))\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad - \left(\frac{e^{ik\varepsilon}}{2}(1 - ik\varepsilon) \underbrace{\int\limits_0^{\pi} \sin(\theta)\,\mathrm{d}\theta}_{=2} \right) \phi(\mathbf{p})
\end{aligned}
$$

Under the $\varphi \in C^2$ on U assumption, $\phi(\mathbf{q})$ satisfies the Lipschitz continuity condition $\| \phi(\mathbf{q}) - \phi(\mathbf{p}) \| \leq C \| \mathbf{q} - \mathbf{p} \| \leq C\varepsilon$ on the compact $B_\varepsilon^{\mathbf{p}}$ for a non-negative constant C.
It follows :

$$
\lim_{\varepsilon \to 0} H_1 = -\phi(\mathbf{p})
$$

$$
\begin{aligned}
H_2 &= -\int\limits_{\partial B_\varepsilon^{\mathbf{p}}} G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q}\,\mathrm{d}q \\
&= -\int\limits_0^{2\pi}\int\limits_0^{\pi}\left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon}\nabla\phi(\mathbf{q})\cdot\mathbf{n}_q\right)\varepsilon^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}(\nabla\phi(\mathbf{q})\cdot\mathbf{n}_q)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}(\nabla\phi(\mathbf{q})-\nabla\phi(\mathbf{p}))\cdot\mathbf{n}_q\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}\nabla\phi(\mathbf{p})\cdot\mathbf{n}_q\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}
$$

Under the $\varphi \in C^2$ on U assumption, $\phi(\mathbf{q})$ also satisfies the Lipschitz continuity condition $\| \nabla\phi(\mathbf{q}) - \nabla\phi(\mathbf{p}) \|_{max} \leq C \| \mathbf{q} - \mathbf{p} \| \leq C\varepsilon$ on the compact $B_\varepsilon^{\mathbf{p}}$ for a non-negative constant C. Therefore:

$$\lim_{\varepsilon\to 0} H_2 = 0$$

(5.1) expressed in terms of H_1, H_2, I_1 and I_2 is:

$$\int\limits_{\partial U_\varepsilon}\left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q = H_1 + H_2 + I_1 + I_2 = 0$$

The above equation rearranged and evaluated for $\lim \varepsilon\to 0$:

$$-H_1 = I_1 + I_2$$

We arrive at the boundary integral formulation for the observation point $\mathbf{p}$ in the interior of U.

$$\phi(\mathbf{p}) = \int\limits_{\partial U}\left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q \tag{5.2}$$

5.2 Observation point on the boundary

Now we consider the case of the observation point $\mathbf{p}$ on the boundary of the domain U and ∂U being smooth in a neighbourhood of $\mathbf{p}$. For $\lim \varepsilon \to 0$ $\partial B_\varepsilon^{\mathbf{p}}$ intersects ∂U. Therefore removing $B_\varepsilon^{\mathbf{p}}$ from U cuts out a circular piece of ∂U. The intersection of U and $\partial B_\varepsilon^{\mathbf{p}}$ is a hemisphere U_{hem} of radius ε. The boundary of the modified domain U_ε can be decomposed into the hemisphere surface ∂U_{hem} and the perforated ∂U, denoted ∂U_{perf}.

$$\partial U_\varepsilon := \partial U_{perf} \,\dot{\cup}\, \partial U_{hem}$$

$$\partial U_{hem} = \{\mathbf{q} \in U \mid \parallel \mathbf{q} - \mathbf{p} \parallel = \varepsilon\}$$

$$\partial U_{perf} = \partial U \setminus \{\mathbf{q} \in U \mid \parallel \mathbf{q} - \mathbf{p} \parallel \leq \varepsilon\ \}$$

Again the two terms of the left-hand side of (5.1) will be examined separately for ∂U_{hem} and ∂U_{perf}. The hemisphere U_{hem} will be represented through spherical coordinates.

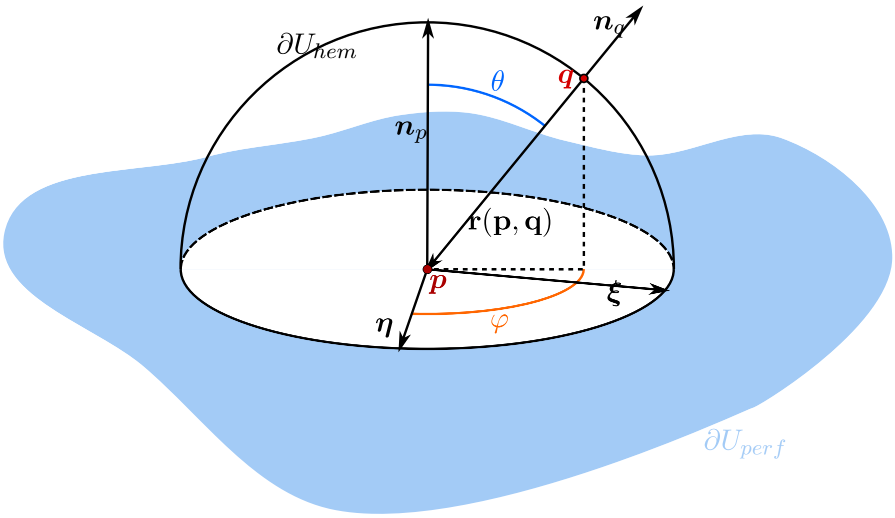

Figure 5.2: The hemisphere at $\mathbf{p}$

We define a local orthonormal base similar to figure 5.1 at $\mathbf{p}$ with the additional requirement that $\boldsymbol{\tau} = \boldsymbol{n}_p$.

A point $\mathbf{q} \in \partial U_{hem}$ in the sperical coordinates of the local orthonormal base:

$$\mathbf{q} = \varepsilon(\sin(\theta)\cos(\varphi)\boldsymbol{\eta} + \sin(\theta)\sin(\varphi)\boldsymbol{\xi} + \cos(\theta)\boldsymbol{n}_p) \tag{5.3}$$

The separate terms of equation (5.1):

$$H_1 := \int\limits_{\partial U_{hem}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q})\,\mathrm{d}q$$

$$H_2 := -\int\limits_{\partial U_{hem}} G(\mathbf{p},\mathbf{q})\frac{\partial \phi(\mathbf{q})}{\partial n_q}\,\mathrm{d}q$$

$$I_1 := \int\limits_{\partial U_{perf}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q})\,\mathrm{d}q$$

$$I_2 := -\int\limits_{\partial U_{perf}} G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \,\mathrm{d}q$$

$$\begin{aligned}
H_1 &= \int\limits_{\partial U_{hem}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) \,\mathrm{d}q \\
&= \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \left(\frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}}_{=1} \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon^2} (ik\varepsilon - 1) \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= -\frac{e^{ik\varepsilon}}{4\pi} (1 - ik\varepsilon) \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} (\phi(\mathbf{q}) - \phi(\mathbf{p})) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&\quad - \frac{e^{ik\varepsilon}}{4\pi} (1 - ik\varepsilon) \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \phi(\mathbf{p}) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= -\frac{e^{ik\varepsilon}}{4\pi} (1 - ik\varepsilon) \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} (\phi(\mathbf{q}) - \phi(\mathbf{p})) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&\quad - \left(\frac{e^{ik\varepsilon}}{2} (1 - ik\varepsilon) \underbrace{\int\limits_0^{\frac{\pi}{2}} \sin(\theta) \,\mathrm{d}\theta}_{=1} \right) \phi(\mathbf{p})
\end{aligned}$$

As above, $\phi(\mathbf{q})$ satisfies the Lipschitz continuity condition $\| \phi(\mathbf{q}) - \phi(\mathbf{p}) \| \leq C \| \mathbf{q} - \mathbf{p} \| \leq C\varepsilon$ on the compact $B_\varepsilon^{\mathbf{p}} \cap U$ for a non-negative constant C. It follows:

$$\lim_{\varepsilon \to 0} H_1 = -\frac{1}{2}\phi(\mathbf{p})$$

$$
\begin{aligned}
H_2 = & -\int\limits_{\partial U_{hem}} G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q}\, \mathrm{d}q \\
= & -\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon} \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \varepsilon^2 \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi \\
= & -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi} \int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \left(\frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi \\
= & -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi} \int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} (\nabla\phi(\mathbf{q}) - \nabla\phi(\mathbf{p})) \cdot \mathbf{n}_q \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi \\
& -\frac{\varepsilon\, e^{ik\varepsilon}}{4\pi} \int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \nabla\phi(\mathbf{p}) \cdot \mathbf{n}_q \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi
\end{aligned}
$$

Again we assert $\phi(\mathbf{q})$ satisfies the Lipschitz continuity condition $\parallel \nabla\phi(\mathbf{q}) - \nabla\phi(\mathbf{p}) \parallel_{max} \leq C \parallel \mathbf{q} - \mathbf{p} \parallel \leq C\varepsilon$ on the compact $B_\varepsilon^{\mathbf{p}} \cap U$ for a non-negative constant C. Therefore:

$$\lim_{\varepsilon\to 0} H_2 = 0$$

Analogously to the previous section, (5.1) is expressed in terms of H_1, H_2, I_1 and I_2:

$$\int\limits_{\partial U_\varepsilon} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q = H_1 + H_2 + I_1 + I_2 = 0$$

The above equation rearranged and evaluated for $\lim \varepsilon\to 0$:

$$-H_1 = I_1 + I_2$$

We arrive at the boundary integral formulation for the observation point $\mathbf{p}$ on the boundary ∂U.

$$\frac{1}{2}\phi(\mathbf{p}) = \int\limits_{\partial U_{perf} = \partial U \setminus \{\mathbf{p}\}} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \tag{5.4}$$

The expression $\partial U \setminus \{\mathbf{p}\}$ is shorthand for $\lim_{\varepsilon\to 0} \partial U \setminus B_\varepsilon^{\mathbf{p}}$.
In the case of the observation point $\mathbf{p}$ on the boundary and between smooth boundary pieces and $\lim \varepsilon\to 0$ the integrals I_1 and I_2 stay the same, but the integration limits of

H_1 and H_2 change. Nevertheless $H_2 = 0$. H_1 takes the following form:

$$
\begin{aligned}
H_1 = & \int\limits_{S:=U\cap\partial B_\varepsilon^{\mathbf{p}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q})\,\mathrm{d}q \\
= & \iint\limits_S \left(\frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}}_{=1} \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
= & \iint\limits_S \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon^2}(ik\varepsilon - 1)\phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
= & -\frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon) \iint\limits_S (\phi(\mathbf{q}) - \phi(\mathbf{p}))\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
& -\frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon) \iint\limits_S \phi(\mathbf{p})\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
= & -\frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon) \iint\limits_S (\phi(\mathbf{q}) - \phi(\mathbf{p}))\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
& - \left(\frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon) \underbrace{\iint\limits_S \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{\Omega(\mathbf{p})} \right) \phi(\mathbf{p}) \\
\implies \lim_{\varepsilon\to 0} H_1 = & -\frac{\Omega(\mathbf{p})}{4\pi}\phi(\mathbf{p})
\end{aligned}
$$

Here $\Omega(\mathbf{p})$ denotes the solid angle of the intersection of $B_\varepsilon^{\mathbf{p}}$ and U. We arrive at the boundary integral formulation for the observation point $\mathbf{p}$ on the boundary ∂U and between smooth boundary pieces (on the edge between).

$$
\frac{\Omega(\mathbf{p})}{4\pi}\phi(\mathbf{p}) = \int\limits_{\partial U_{perf}=\partial U\setminus\{\mathbf{p}\}} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \tag{5.5}
$$

5.3 Field modification

So far we have only considered an acoustic field defined by its boundary and boundary condition alone. In this section we extend the boundary equation to model an existing sound field, which is modified by the boundary. At first we address the problem of an acoustic field ϕ generated in a domain U_ε due to a point source $G(\mathbf{q},\mathbf{s})$ at the interior position $\mathbf{s} \in U_\varepsilon^{\mathrm{o}}$.

The unmodified point source field is described via the free-field Greens function.

$$G(\mathbf{q},\mathbf{s}) = \frac{1}{4\pi}\frac{e^{ik\|\mathbf{q}-\mathbf{s}\|}}{\|\mathbf{q}-\mathbf{s}\|} = \frac{1}{4\pi}\frac{e^{ikr}}{r}$$

$$r = \|\mathbf{r}\| = \|\mathbf{q}-\mathbf{s}\|$$

It is assumed that near $\mathbf{s}$ the modified field is similar to the free-field, which means that the difference $\phi_{diff}(\mathbf{q}) \coloneqq \phi(\mathbf{q}) - G(\mathbf{q},\mathbf{s})$ between the modified field and the free-field solution and also its gradient $\nabla\phi_{diff}(\mathbf{q})$ are continuous in a neighbourhood of $\mathbf{s}$ [CL07, page 13].

This implies ϕ being singular at $\mathbf{s}$ and prevents us again from applying Greens second theorem directly to $\phi(\mathbf{q})$ and $G(\mathbf{p},\mathbf{q})$ on U_ε. Similar to above we modify U_ε by defining a ball $B_\varepsilon^{\mathbf{s}}$ of radius $\varepsilon\to 0$ around the source location $\mathbf{s}$ and removing it from the domain.

$$B_\varepsilon^{\mathbf{s}} \coloneqq \{\mathbf{q}\in U \mid \|\mathbf{q}-\mathbf{s}\| < \varepsilon\}$$

The modified domain is now:

$$U_s = U_\varepsilon \setminus B_\varepsilon^{\mathbf{s}} = U_\varepsilon \setminus \{\mathbf{q}\in U \mid \|\mathbf{q}-\mathbf{s}\| < \varepsilon\}$$

The boundary of U_s writes as:

$$\partial U_s = \partial U_\varepsilon \,\dot\cup\, \partial B_\varepsilon^{\mathbf{s}}$$

As Greens second identity applies to ∂U_s:

$$\begin{aligned}\int\limits_{\partial U_s}\left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q}\right)\mathrm{d}q &= \int\limits_{U_s}\left(G(\mathbf{p},\mathbf{q})\,\nabla^2\phi(\mathbf{q}) - \nabla^2 G(\mathbf{p},\mathbf{q})\,\phi(\mathbf{q})\right)\mathrm{d}q \\ &= \int\limits_{U_s} G(\mathbf{p},\mathbf{q})\underbrace{(\nabla^2+k^2)\,\phi(\mathbf{q})}_{=0}\,\mathrm{d}q \\ &\quad - \int\limits_{U_s}\underbrace{(\nabla^2+k^2)\,G(\mathbf{p},\mathbf{q})}_{=0}\,\phi(\mathbf{q})\,\mathrm{d}q \\ &= 0\end{aligned} \tag{5.6}$$

The surface integrals over $\partial B_\varepsilon^{\mathbf{s}}$ will also be evaluated separately. We employ a similar coordinate transform for $\partial B_\varepsilon^{\mathbf{s}}$ as for $\partial B_\varepsilon^{\mathbf{p}}$, see figure 5.1 for reference.

$$S_1 \coloneqq \int\limits_{\partial B_\varepsilon^{\mathbf{s}}}\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q})\,\mathrm{d}q$$

$$S_2 \coloneqq -\int\limits_{\partial B_\varepsilon^{\mathbf{s}}} G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q}\,\mathrm{d}q$$

$$
\begin{aligned}
|S_1| =& \left| \int\limits_{\partial B_\varepsilon^{\mathbf{s}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) \,\mathrm{d}q \right| \\
=& \left| \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right| \\
=& \left| \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) \phi(\mathbf{q}) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right| \\
\leq& \varepsilon^2 \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1)(\phi(\mathbf{q}) - G(\mathbf{q},\mathbf{s})) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right| \\
&+ \varepsilon^2 \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) G(\mathbf{q},\mathbf{s}) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right| \\
=& \varepsilon^2 \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) \phi_{diff}(\mathbf{q}) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right| \\
&+ \varepsilon^2 \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) \frac{e^{ik\varepsilon}}{4\pi\varepsilon} \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right| \\
\leq& \varepsilon^2 \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) \phi_{diff}(\mathbf{q}) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right| \\
&+ \varepsilon \left| \frac{e^{ik\varepsilon}}{16\pi^2} \right| \left| \left(\int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{r(\mathbf{p},\mathbf{q})^2} (ikr(\mathbf{p},\mathbf{q}) - 1) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \right) \right|
\end{aligned}
$$

ϕ_{diff} is continuous and attains its maximum and minimum on $B_\varepsilon^{\mathbf{s}}$ for small ε. Also $\lim_{\varepsilon \to 0} r(\mathbf{p},\mathbf{q}) = \| \mathbf{p} - \mathbf{s} \|$ for all $\mathbf{q} \in \partial B_\varepsilon^{\mathbf{s}}$. It follows:

$$
\lim_{\varepsilon \to 0} S_1 = 0
$$

$$
\begin{aligned}
S_2 = &- \int\limits_{\partial B_\varepsilon^{\mathbf{s}}} G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\
= &- \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \nabla \phi(\mathbf{q}) \cdot \mathbf{n}_q \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
= &- \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \nabla (\phi(\mathbf{q}) - G(\mathbf{q},\mathbf{s})) \cdot \mathbf{n}_q \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&- \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \frac{\partial G(r(\mathbf{q},\mathbf{s}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{q},\mathbf{s})}{\partial n_q}}_{=1} \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
= &- \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \nabla \phi_{diff}(\mathbf{q}) \cdot \mathbf{n}_q \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&- \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \frac{e^{ik\varepsilon}}{4\pi\varepsilon^2} (ik\varepsilon - 1) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi
\end{aligned}
$$

Under the continuity assumption for $\nabla \phi_{diff}$ it follows for the first term of S_2:

$$
\lim_{\varepsilon \to 0} S_{21} = 0
$$

And for the second term of S_2:

$$
\begin{aligned}
\lim_{\varepsilon \to 0} S_{22} = &\lim_{\varepsilon \to 0} - \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})} \frac{e^{ik\varepsilon}}{4\pi\varepsilon^2} (ik\varepsilon - 1) \right) \varepsilon^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
= &\frac{1}{4\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\frac{e^{ikr(\mathbf{p},\mathbf{s})}}{4\pi r(\mathbf{p},\mathbf{s})} \right) \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
= &\frac{1}{4\pi} \frac{e^{ikr(\mathbf{p},\mathbf{s})}}{4\pi r(\mathbf{p},\mathbf{s})} \int\limits_0^{2\pi} \int\limits_0^{\pi} \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
= &\frac{e^{ikr(\mathbf{p},\mathbf{s})}}{4\pi r(\mathbf{p},\mathbf{s})} \\
= &G(\mathbf{p},\mathbf{s})
\end{aligned}
$$

(5.6) expressed in terms of H_1, H_2, I_1, I_2, S_1 and S_2 is:

$$
\int\limits_{\partial U_s} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q = H_1 + H_2 + I_1 + I_2 + S_1 + S_2 = 0
$$

The above equation rearranged and evaluated for $\lim \varepsilon \to 0$:

$$-H_1 = I_1 + I_2 + S_2 = I_1 + I_2 + G(\mathbf{p}, \mathbf{s}) \tag{5.7}$$

Expression (5.7) implies that an incident point source field, which gets modified (reflected, difffracted and absorbed) by boundaries, just adds a source term as observed at the obervation point $\mathbf{p}$ in the free field case to the boundary equation. In the case of multiple point sources, the source term is the sum of the sources as observed at $\mathbf{p}$. By the virtue of Huygens principle a complex sound field can be modeled by a suitable amount of point sources. Therefore the source term in (5.7) can be generalized to the case of any incident field in the domain, that would exist if there were no boundaries. This result is in agreement with [Juh94, page 22]. In the ensuing chapters the incident source term oberved at $\mathbf{p}$ will be labeled $\phi^{in}(\mathbf{p})$.

5.4 Boundary equations with incident field

The integral equations thus far with an additional source term:
The boundary integral equation for the observation point $\mathbf{p}$ in the interior U° of the domain U.

$$\phi(\mathbf{p}) = \phi^{in}(\mathbf{p}) + \int\limits_{\partial U} \left(\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \tag{5.8}$$

The boundary integral equation for the observation point $\mathbf{p}$ on the boundary ∂U of the domain U, where the boundary is smooth in a neighbourhood of $\mathbf{p}$.

$$\frac{1}{2}\phi(\mathbf{p}) = \phi^{in}(\mathbf{p}) + \int\limits_{\partial U} \left(\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \tag{5.9}$$

The boundary integral equation for the observation point $\mathbf{p}$ on the boundary ∂U of the domain U, where $\mathbf{p}$ lies between smooth boundary pieces.

$$\frac{\Omega(\mathbf{p})}{4\pi}\phi(\mathbf{p}) = \phi^{in}(\mathbf{p}) + \int\limits_{\partial U} \left(\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p}, \mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \tag{5.10}$$

Note that for equations with $\mathbf{p} \in \partial U$, we define $\partial U :=_{\lim \varepsilon \to 0} \partial U \backslash B_{\varepsilon}^{\mathbf{p}}$.

6 Numerical solution of the boundary equations

So far we have transformed our partial differential equation into a boundary integral. The boundary integral equation is generally not easier to solve analytically than the partial differential equation. Only for very simple boundary geometries is an analytical solution viable. This chapter deals with the necessary reduction of the boundary equation to a system of linear equations. A rather straightforward approach is introduced, the collocation method, which demands that the integral equation be satisfied for a discrete set of points on the boundary. We will pursue a technique whereby the boundary is decomposed into/approximated by n non-intersecting triangular panels. We'll enforce that the integral equation be satisfied at the centroids of the triangles, the collocation points. Another approximation will be conducted on the boundary solutions, the sound pressure ϕ and the surface particle velocity $\frac{\partial \phi}{\partial n}$, by assuming them as constant over an element. These approximations of the boundary shape and the boundary values will introduce an error, which will generally tend to zero as the element size goes to zero also [Juh94, page 40].

From now on the surface particle velocity $\frac{\partial \phi}{\partial n}$ will also be denoted v.

6.1 Approximation of the boundary

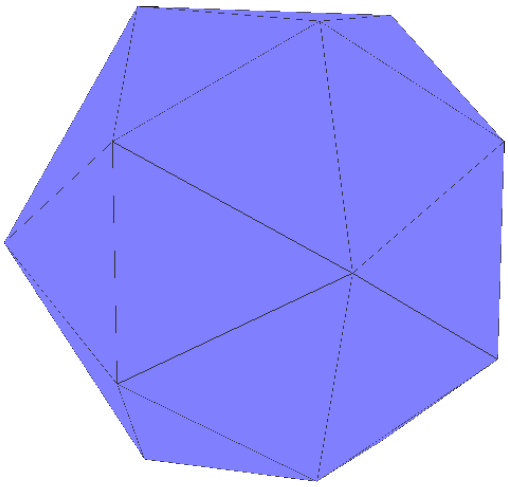

Figure 6.1: Rough approximation of a sphere with 36 triangles

As outlined above, the following text deals with a numerical integration method, with triangles as boundary representation. ∂U shall denote our original boundary and $\triangle U_i$ (for $i = 1, 2, \dots n$) shall be the approximating triangular panels.

$$\partial U \approx \partial \widetilde{U} = \dot{\bigcup_{i=1 \dots n}} \triangle U_i$$

As the collocation points desribed above are situated on the boundary on the interior of a planar element, (5.9) is the appropriate equation to identify the solution at a collocation point with the boundary integral. Let $\mathbf{p}_i$ be the collocation point on $\triangle U_i$. Applying (5.9) to the triangular boundary leads to:

$$\begin{aligned} \frac{1}{2}\phi(\mathbf{p}_i) &= \phi^{in}(\mathbf{p}_i) + \int\limits_{\partial \widetilde{U}} \left(\frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p}_i, \mathbf{q}) \upsilon(\mathbf{q}) \right) \mathrm{d}q \\ &= \phi^{in}(\mathbf{p}_i) + \sum_{j=1}^{n} \int\limits_{\triangle U_j} \left(\frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q} \phi(\mathbf{q}) - G(\mathbf{p}_i, \mathbf{q}) \upsilon(\mathbf{q}) \right) \mathrm{d}q \end{aligned} \tag{6.1}$$

6.2 The system of linear equations

Assuming constant behaviour of the boundary state functions ϕ and υ for a triangle $\triangle U_l$ suggests indexing the state functions the same as their triangular domain.

$$\phi_l = \phi(\mathbf{q}) \quad \forall \mathbf{q} \in \triangle U_l$$

$$\upsilon_l = \upsilon(\mathbf{q}) \quad \forall \mathbf{q} \in \triangle U_l$$

$$\begin{aligned} \frac{1}{2}\phi(\mathbf{p}_i) &= \phi^{in}(\mathbf{p}_i) + \sum_{j=1}^{n} \int\limits_{\triangle U_j} \left(\frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q} \phi_j - G(\mathbf{p}_i, \mathbf{q}) \upsilon_j \right) \mathrm{d}q \\ &= \phi^{in}(\mathbf{p}_i) + \sum_{j=1}^{n} \int\limits_{\triangle U_j} \frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q} \,\mathrm{d}q\, \phi_j - \sum_{j=1}^{n} \int\limits_{\triangle U_j} G(\mathbf{p}_i, \mathbf{q}) \,\mathrm{d}q\, \upsilon_j \end{aligned}$$

Above equation can be rearranged to:

$$\sum_{j=1}^{n} \int\limits_{\triangle U_j} G(\mathbf{p}_i, \mathbf{q}) \,\mathrm{d}q \, \upsilon_j = \phi^{in}(\mathbf{p}_i) + \sum_{j=1}^{n} \left(\int\limits_{\triangle U_j} \frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q} \,\mathrm{d}q - \delta_{ij} \frac{1}{2} \right) \phi_j, \tag{6.2}$$

where δ_{ij} is the Kronecker delta. Ordering the respective equations for the n collocation points leads directly to the following system of linear equations.

$$\begin{pmatrix} \int\limits_{\triangle U_1} G(\mathbf{p}_1, \mathbf{q}) \,\mathrm{d}q & \dots & \int\limits_{\triangle U_n} G(\mathbf{p}_1, \mathbf{q}) \,\mathrm{d}q \\ \vdots & \ddots & \vdots \\ \int\limits_{\triangle U_1} G(\mathbf{p}_n, \mathbf{q}) \,\mathrm{d}q & \dots & \int\limits_{\triangle U_n} G(\mathbf{p}_n, \mathbf{q}) \,\mathrm{d}q \end{pmatrix} \begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_n \end{pmatrix}$$

$$= \begin{pmatrix} \phi^{in}(\mathbf{p}_1) \\ \vdots \\ \phi^{in}(\mathbf{p}_n) \end{pmatrix} + \left(\begin{pmatrix} \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \dots & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\ \vdots & \ddots & \vdots \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \dots & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_q} \,\mathrm{d}q \end{pmatrix} - \frac{1}{2} I \right) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

I is the $n \times n$ identity matrix.
Due to the boundary conditions, we can substitute ϕ_i into υ_i or vice versa.

$$a_i \phi_i + b_i \upsilon_i = f_i \quad \text{for } i = 1, \dots, n$$

Therefore we can transform the above general linear system into a $A\mathbf{b} = \mathbf{c}$ standard form, with $A \in \mathbb{C}^{n \times n}$ and $\mathbf{b}$, $\mathbf{c} \in \mathbb{C}^n$.

6.3 Solving the discrete integrals

For each of the collocation points, we have to solve the integrals of $\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}$ and $G(\mathbf{p}, \mathbf{q})$ over each triangular panel. When $\mathbf{p}$ does not lie on the triangle, the integrand is continuously differentiable, hence we can evaluate the integral directly via a Gauss quadrature rule [Kir98, page 40]. When $\mathbf{p}$ lies on the panel, the Gauss quadrature converges only very slowly, due to the singularity at $\mathbf{p} = \mathbf{q}$. In this section we show the derivation of a nonsingular analytical reformulation of the following singular integrals after a manner outlined in [MZHT10, page 198]:

$$I_1 := \lim_{\varepsilon \to 0} \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \,\mathrm{d}q$$

$$I_2 := \lim_{\varepsilon \to 0} \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} G(\mathbf{p}, \mathbf{q}) \,\mathrm{d}q$$

Note that the normal $\mathbf{n}$ to a planar triangle is constant and perpendicular to $\mathbf{r} = \mathbf{p} - \mathbf{q}$. Therefore $\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} = -\frac{\mathbf{r}\cdot\mathbf{n}_q}{\|\mathbf{r}\|} = 0$. It follows

$$I_1 = \lim_{\varepsilon \to 0} \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} \frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q}}_{=0} \mathrm{d}q$$
$$=0$$

I_2 is evaluated via a transformation to cylindrical coordinates.

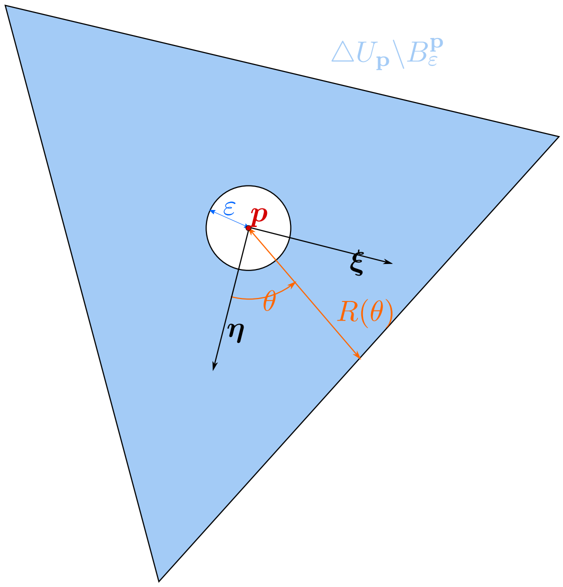

Figure 6.2: The triangle $\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}$

The unit vectors $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and $\boldsymbol{n}_p$ form a local orthogonal base like in figure 5.2. $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are tangential to the triangle. A point $\mathbf{q} \in \triangle U_{\mathbf{p}}$ in the cylindrical coordinates of the orthonormal base of $B_\varepsilon^{\mathbf{p}}$:

$$\mathbf{q} = r(\mathbf{p},\mathbf{q})\cos(\theta)\boldsymbol{\eta} + r(\mathbf{p},\mathbf{q})\sin(\theta)\boldsymbol{\xi} + 0\boldsymbol{n}_p \tag{6.3}$$

A point $\mathbf{q}$ on the edge of $\triangle U_{\mathbf{p}}$ in the cylindrical coordinates of the orthonormal base of $B_\varepsilon^{\mathbf{p}}$:

$$\mathbf{q} = R(\theta)\cos(\theta)\boldsymbol{\eta} + R(\theta)\sin(\theta)\boldsymbol{\xi} \tag{6.4}$$

Note that the Jacobian determinant of the transformation from the original cartesian coordinates to the cylindrical coordinates is $r = r(\mathbf{p},\mathbf{q})$.

$$
\begin{aligned}
I_2 &= \lim_{\varepsilon\to 0} \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} G(\mathbf{p},\mathbf{q})\,\mathrm{d}q \\
&= \lim_{\varepsilon\to 0} \int\limits_0^{2\pi} \int\limits_\varepsilon^{R(\theta)} G(r)\, r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \lim_{\varepsilon\to 0} \int\limits_0^{2\pi} \int\limits_\varepsilon^{R(\theta)} \frac{e^{ikr}}{4\pi r}\, r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int\limits_0^{2\pi} -\frac{i}{4\pi k} \left[e^{ikr}\right]_0^{R(\theta)} \mathrm{d}\theta \\
&= \int\limits_0^{2\pi} \frac{i}{4\pi k} \left(1 - e^{ikR(\theta)}\right) \mathrm{d}\theta \\
&= \frac{i}{2k} + \frac{i}{4\pi k} \int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta
\end{aligned}
$$

The remaining integral term is nonsingular and can be directly evaluated with a standard gaussian quadrature rule.

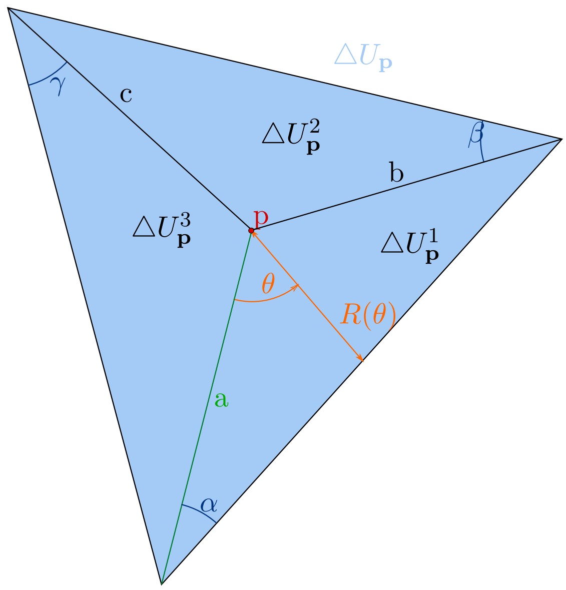

Figure 6.3: $R(\theta)$ on the triangle $\triangle U_{\mathbf{p}}$

Due to the law of sines, $R(\theta)$ can be exemplarily evaluated on $\triangle U^1_{\mathbf{p}}$ the following way:

$$R(\theta) = a\frac{sin(\alpha)}{sin(\pi - \alpha - \theta)} = a\frac{sin(\alpha)}{sin(\alpha + \theta)}$$

Finally the boundary equation (6.2) with only nonsingular integrals:

$$\left(\frac{i}{2k} + \frac{i}{4\pi k}\int\limits_{0}^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta\right)\upsilon_i + \sum_{\substack{j=1\\ j\neq i}}^{n}\int\limits_{\triangle U_j} G(\mathbf{p}_i, \mathbf{q})\,\mathrm{d}q\,\upsilon_j = \phi^{in}(\mathbf{p}_i) - \frac{1}{2}\phi_i + \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\int\limits_{\triangle U_j}\frac{\partial G(\mathbf{p}_i, \mathbf{q})}{\partial n_q}\,\mathrm{d}q\right)\phi_j \tag{6.5}$$

6.4 Numerical and inherent shortcomings

In this section we will list some shortcomings of the boundary element method resulting from (5.9). In the interior case with Dirichlet boundary conditions equation (5.9) turns into a Fredholm integral equation of the first kind, which are generally difficult to solve. The matrices that arise in their equivalent linear systems are ill-conditioned [Kir98, page 57].

The general boundary element method for the *exterior* Helmholtz problem suffers from the so-called *non-uniqueness problem.* Regardless of the imposed boundary conditions, be it Dirichlet, Neumann or impedance, the Helmholtz integral equation (5.9) does not yield a unique solution at the eigenfrequencies of the corresponding interior Dirichlet problem [MW08, page 412]. Though physically unconnected, a resonance in the interior problem pollutes the solution of the exterior problem. These characteristic frequencies of the interior Dirichlet are therefore also termed *fictitious* eigenfrequencies. The *non-uniqueness problem* is not due to a specific numerical implementation, but inherent to the integral formulation. The interior problem itself is not affected by this issue as the eigenfrequencies of the interior problem are the 'real' eigenfrequencies of the interior domain [Juh94, page 106]. It is generally not a feasible strategy to try to avoid these contaminating wavenumbers, as they become closer spaced at higher frequencies and also render the matrix of the coresponding linear system ill-conditioned for neighbouring wavenumbers [Kir98, page 57]. The approach to solve the interior Dirichlet problem first, to vet a specific wavenumber is further problematic, because of the Fredholm first kind equation difficulty mentioned above. Therefore improved boundary integral equation formulation were explored by the researchers in the field. The most popular approaches are the CHIEF and the Burton-Miller method. The CHIEF method is just mentioned here for completeness. This text focuses on the Burton-Miller method in the following.

7 The Burton-Miller formulation

In their 1971 article [BM71], Burton and Miller derive a second boundary integral equation for the observation point on the boundary, by differentiating (5.9) with regard to the normal $\mathbf{n}_p$ to the boundary at $\mathbf{p}$. Note that (5.9) is a function of $\mathbf{p}$. Therefore the directional derivative of the boundary equation is the following:

$$\frac{1}{2}\upsilon(\mathbf{p}) = \frac{1}{2}\frac{\partial\phi(\mathbf{p})}{\partial n_p} = \frac{\partial\phi^{in}(\mathbf{p})}{\partial n_p} + \int\limits_{\partial U} \left(\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}\phi(\mathbf{q}) - \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p}\upsilon(\mathbf{q}) \right) \mathrm{d}q \tag{7.1}$$

The further approach is to combine (5.9) and (7.1) as a linear combination. The Burton-Miller combined boundary equation [CCH09, page 165]:

$$\begin{aligned} \frac{1}{2}\phi(\mathbf{p}) + \alpha\frac{1}{2}\upsilon(\mathbf{p}) =& \phi^{in}(\mathbf{p}) + \alpha\frac{\partial\phi^{in}(\mathbf{p})}{\partial n_p} \\ &+ \int\limits_{\partial U} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\phi(\mathbf{q}) - G(\mathbf{p},\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial n_q} \right) \mathrm{d}q \\ &+ \alpha\int\limits_{\partial U} \left(\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}\phi(\mathbf{q}) - \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p}\upsilon(\mathbf{q}) \right) \mathrm{d}q \end{aligned} \tag{7.2}$$

with coupling parameter $\alpha \in \mathbb{C}$.
Most of the encountered literature gives $\alpha = \frac{i}{k}$ as an optimal choice for $e^{-i\omega t}$-time dependence in (4.2) (our case) and $\alpha = -\frac{i}{k}$ for $e^{i\omega t}$-time dependence. See [ZCGD15, page 50] for reference. We will return to the problem of the optimal coupling parameter in the conclusion section.

7.1 Solving the singular integrals

We employ the same collocation scheme for the Burton-Miller combined boundary equation (7.2) as for our original boundary equation (5.9). The normal derivatives of $G(\mathbf{p},\mathbf{q})$ in equation (7.1) have already been given in section (4.6.3). Those derivatives are also singular on $\triangle U_{\mathbf{p}}$ and, similar to the situation in section (6.3), not eligible for Gaussian quadrature. In this chapter we reproduce the derivation of nonsingular reformulations of the $(\triangle U_{\mathbf{p}} \backslash \{\mathbf{p}\})$-integrals for planar triangles with constant ϕ and υ, originally from [MZHT10, page 198].

$$I_1 := \int\limits_{\triangle U_{\mathbf{p}} \backslash B_\varepsilon^{\mathbf{p}}} \frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}\phi(\mathbf{q})\,\mathrm{d}q$$

$$I_2 \coloneqq - \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{P}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p} \upsilon(\mathbf{q}) \, \mathrm{d}q$$

It holds:

$$\begin{aligned}
\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p} =& \frac{\partial^2 G(r(\mathbf{p},\mathbf{q}))}{\partial r^2} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} + \frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \frac{\partial^2 r(\mathbf{p},\mathbf{q})}{\partial n_p \partial n_q} \\
=& \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^3} \left((3 - 3ikr(\mathbf{p},\mathbf{q}) - k^2 r(\mathbf{p},\mathbf{q})^2) \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} + (1 - ikr(\mathbf{p},\mathbf{q}))\, \mathbf{n}_p \cdot \mathbf{n}_q \right)
\end{aligned} \tag{7.3}$$

The vectors $r(\mathbf{p},\mathbf{q})$ and $\mathbf{n}_q = \mathbf{n}_p$ are orthogonal on $\triangle U_{\mathbf{p}}$. It follows:

$$\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} = \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} = 0 \tag{7.4}$$

(7.4) reduces $\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}$ to:

$$\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p} = \frac{e^{ikr(\mathbf{p},\mathbf{q})}}{4\pi r(\mathbf{p},\mathbf{q})^3} (1 - ikr(\mathbf{p},\mathbf{q}))$$

I_1 is evaluated now via the same transform to cylindrical coordinates as described by figure 6.2, (6.3) and (6.4). Note that the assumption of constant behaviour for ϕ on the triangle is used in the following.

$$\begin{aligned}
I_1 &= \int\limits_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{P}}} \frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p} \phi(\mathbf{q}) \, \mathrm{d}q \\
&= \left(\int\limits_0^{2\pi} \int\limits_\varepsilon^{R(\theta)} \frac{e^{ikr}}{4\pi r^3} (1 - ikr)\, r \, \mathrm{d}r \, \mathrm{d}\theta \right) \phi(\mathbf{p}) \\
&= \left(\int\limits_0^{2\pi} \left[-\frac{e^{ikr}}{4\pi r} \right]_\varepsilon^{R(\theta)} \mathrm{d}\theta \right) \phi(\mathbf{p}) \\
&= \left(\int\limits_0^{2\pi} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon} - \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \right) \mathrm{d}\theta \right) \phi(\mathbf{p}) \\
&= \left(\frac{e^{ik\varepsilon}}{2\varepsilon} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \mathrm{d}\theta \right) \phi(\mathbf{p})
\end{aligned}$$

Under abuse of notation we can therefore write:

$$\lim_{\varepsilon \to 0} I_1 = \left(\frac{1}{2\varepsilon} - \int_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \, \mathrm{d}\theta \right) \phi(\mathbf{p})$$

Note that I_1 includes a divergent term, proportional to $\frac{1}{\varepsilon}$.

$$\begin{aligned} I_2 &= - \int_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p} \upsilon(\mathbf{q}) \, \mathrm{d}q \\ &= - \int_{\triangle U_{\mathbf{p}} \setminus B_\varepsilon^{\mathbf{p}}} \frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \underbrace{\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p}}_{=0} \upsilon(\mathbf{q}) \, \mathrm{d}q \\ &= 0 \end{aligned}$$

Because I_1 includes an ε-dependent term, we have to revaluate the hemisphere surface integrals.

$$\partial U_{hem} = \partial(B_\varepsilon^{\mathbf{p}} \cap \widetilde{U})$$

$$H_1 := - \int_{\partial U_{hem}} \frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p} \phi(\mathbf{q}) \, \mathrm{d}q$$

$$H_2 := \int_{\partial U_{hem}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p} \upsilon(\mathbf{q}) \, \mathrm{d}q$$

The signs of the integrals above are changed with regard to the corresponding integrals of section (5.2), because in the derivation of equation (7.1) the integrals have been moved to the other side of the equality sign. Note that the hemisphere surface is identical to the definition illustrated by figure 5.2. Again we transform the hemisphere surface to spherical coordinates. A point $\mathbf{q} \in \partial U_{hem}$ in the spherical coordinates of the local orthonormal base:

$$\mathbf{q} = \varepsilon(\sin(\theta)\cos(\varphi)\boldsymbol{\eta} + \sin(\theta)\sin(\varphi)\boldsymbol{\xi} + \cos(\theta)\boldsymbol{n}_p) \tag{7.5}$$

The following properties hold on the hemisphere:

$$\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_q} = -\frac{\mathbf{r}(\mathbf{p},\mathbf{q}) \cdot \mathbf{n}_q}{\| \mathbf{r}(\mathbf{p},\mathbf{q}) \|} = \mathbf{n}_q \cdot \mathbf{n}_q = 1 \tag{7.6}$$

$$\frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} = \frac{\mathbf{r}(\mathbf{p},\mathbf{q}) \cdot \mathbf{n}_p}{\| \mathbf{r}(\mathbf{p},\mathbf{q}) \|} = -\mathbf{n}_q \cdot \mathbf{n}_p = -\cos(\theta) \tag{7.7}$$

$$r(\mathbf{p},\mathbf{q}) = \varepsilon \tag{7.8}$$

Inserting (7.6), (7.7) and (7.8) into (7.3) lets us express $\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}$ in the following way:

$$\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p} = \frac{e^{ik\varepsilon}}{4\pi\varepsilon^3}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\cos(\theta) \tag{7.9}$$

$\nabla\phi(\mathbf{p})$ in the coordinates of the orthonormal base of the hemisphere:

$$\nabla\phi(\mathbf{p}) = \begin{pmatrix} \frac{\partial\phi(\mathbf{p})}{\partial\eta} \\ \frac{\partial\phi(\mathbf{p})}{\partial\xi} \\ \frac{\partial\phi(\mathbf{p})}{\partial n_p} \end{pmatrix} \tag{7.10}$$

$\mathbf{n}_q$ in the spherical coordinates of the orthonormal base of the hemisphere:

$$\boldsymbol{n}_q = \sin(\theta)\cos(\varphi)\boldsymbol{\eta} + \sin(\theta)\sin(\varphi)\boldsymbol{\xi} + \cos(\theta)\boldsymbol{n}_p \tag{7.11}$$

Still $\phi(\mathbf{q})$ satisfies the Lipschitz continuity condition
$\| \nabla\phi(\mathbf{q}) - \nabla\phi(\mathbf{p}) \|_{max} \leq C \| \mathbf{q} - \mathbf{p} \| \leq C\varepsilon$ on the compact $B_\varepsilon^{\mathbf{p}} \cap U$ for a non-negative constant C. It follows for $\phi(\mathbf{q})$, according to [Yas15, Lemma 1]:

$$\| \phi(\mathbf{q}) - \phi(\mathbf{p}) + \nabla\phi(\mathbf{p}) \cdot \mathbf{r}(\mathbf{p},\mathbf{q}) \| \leq C\varepsilon^2 \tag{7.12}$$

Note that the direction of $\mathbf{r}(\mathbf{p},\mathbf{q})$ is the opposite with regard to the given reference. Above considerations are used in the evaluation of H_1:

$$\begin{aligned}
H_1 =& -\int\limits_{\partial U_{hem}} \frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}\phi(\mathbf{q})\,\mathrm{d}q \\
=& -\int\limits_{\partial U_{hem}} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon^3}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\cos(\theta)\right)\phi(\mathbf{q})\,\mathrm{d}q \\
=& -\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \left(\frac{e^{ik\varepsilon}}{4\pi\varepsilon^3}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\cos(\theta)\right)\phi(\mathbf{q})\varepsilon^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
=& -\frac{e^{ik\varepsilon}}{4\pi\varepsilon}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \left(\cos(\theta)\sin(\theta)\phi(\mathbf{q})\right)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
=& -\frac{e^{ik\varepsilon}}{4\pi\varepsilon}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\left(\underbrace{\phi(\mathbf{q}) - \phi(\mathbf{p}) + \nabla\phi(\mathbf{p})\cdot\mathbf{r}(\mathbf{p},\mathbf{q})}_{\leq C\varepsilon^2}\right)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
& -\frac{e^{ik\varepsilon}}{4\pi\varepsilon}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\phi(\mathbf{p})\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
& -\frac{e^{ik\varepsilon}}{4\pi\varepsilon}(-2 + 2ik\varepsilon + k^2\varepsilon^2)\int\limits_0^{2\pi}\int\limits_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\left(\frac{\partial\phi(\mathbf{p})}{\partial n_q}\varepsilon\right)\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}$$

Note that $\nabla\phi(\mathbf{p})\cdot\mathbf{r} = -\frac{\partial\phi(\mathbf{p})}{\partial n_q}\varepsilon$. We will deal with the three terms of the right-hand side separately. Due to (7.12) the first term $H_{1\,1}$ will tend to zero as $\lim \varepsilon\to 0$.

$$\begin{aligned}
H_{1\,2} =& -\frac{e^{ik\varepsilon}}{4\pi\varepsilon}(-2+2ik\varepsilon+k^2\varepsilon^2)\left(\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos(\theta)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\right)\phi(\mathbf{p}) \\
=& \frac{e^{ik\varepsilon}}{4\varepsilon}(2-2ik\varepsilon-k^2\varepsilon^2)\,\phi(\mathbf{p}) \\
\Longrightarrow \quad \lim_{\varepsilon\to 0} H_{1\,2} =& \frac{1}{2\varepsilon}\phi(\mathbf{p}) - \frac{ik}{2}\phi(\mathbf{p})
\end{aligned} \tag{7.13}$$

Under abuse of notation we can write again:

$$\lim_{\varepsilon\to 0} H_{1\,2} = \frac{1}{2\varepsilon}\phi(\mathbf{p}) - \frac{ik}{2}\phi(\mathbf{p}) \tag{7.14}$$

Note that $H_{1\,2}$ includes an identical divergent term as I_1. The two divergent terms cancel each other out.

$$\begin{aligned}
H_{1\,3} =& -\frac{e^{ik\varepsilon}}{4\pi}(-2+2ik\varepsilon+k^2\varepsilon^2)\left(\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos(\theta)\sin(\theta)\nabla\phi(\mathbf{p})\cdot\mathbf{n}_q\,\mathrm{d}\theta\,\mathrm{d}\varphi\right) \\
\overset{(7.10)\ \&\ (7.11)}{=} & -\frac{e^{ik\varepsilon}}{4\pi}(-2+2ik\varepsilon+k^2\varepsilon^2)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos(\theta)\sin^2(\theta)\cos(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{=0}\right)\frac{\partial\phi(\mathbf{p})}{\partial\eta} \\
& -\frac{e^{ik\varepsilon}}{4\pi}(-2+2ik\varepsilon+k^2\varepsilon^2)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos(\theta)\sin^2(\theta)\sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{=0}\right)\frac{\partial\phi(\mathbf{p})}{\partial\xi} \\
& -\frac{e^{ik\varepsilon}}{4\pi}(-2+2ik\varepsilon+k^2\varepsilon^2)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{\frac{2\pi}{3}}\right)\frac{\partial\phi(\mathbf{p})}{\partial n_p} \\
=& \frac{e^{ik\varepsilon}}{3}(1-1ik\varepsilon-\frac{1}{2}k^2\varepsilon^2)\,\frac{\partial\phi(\mathbf{p})}{\partial n_p} \\
\Longrightarrow \quad \lim_{\varepsilon\to 0} H_{1\,3} =& \frac{1}{3}\frac{\partial\phi(\mathbf{p})}{\partial n_p} = \frac{1}{3}\upsilon(\mathbf{p})
\end{aligned}$$

$$
\begin{aligned}
H_2 &= \int\limits_{\partial U_{hem}} \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p} \frac{\partial \phi(\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\
&= \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \frac{\partial G(r(\mathbf{p},\mathbf{q}))}{\partial r} \frac{\partial r(\mathbf{p},\mathbf{q})}{\partial n_p} \frac{\partial \phi(\mathbf{q})}{\partial n_q} \varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \frac{e^{ik\varepsilon}}{4\pi\varepsilon^2} (ik\varepsilon - 1)\cos(\theta) \nabla\phi(\mathbf{q}) \cdot \mathbf{n}_q \,\varepsilon^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= -\int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \frac{e^{ik\varepsilon}}{4\pi} (ik\varepsilon - 1)\cos(\theta) \underbrace{(\nabla\phi(\mathbf{q}) - \nabla\phi(\mathbf{p})) \cdot \mathbf{n}_q}_{\leq C\varepsilon} \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad - \int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{2}} \frac{e^{ik\varepsilon}}{4\pi} (ik\varepsilon - 1)\cos(\theta) \nabla\phi(\mathbf{p}) \cdot \mathbf{n}_q \,\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}
$$

We will deal with the two terms on the right-hand side separately. The first term $H_{2\,1}$ will tend to zero as $\lim \varepsilon \to 0$.

$$
\begin{aligned}
H_{2\,2} =& -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{e^{ik\varepsilon}}{4\pi}(ik\varepsilon - 1)\cos(\theta)\nabla\phi(\mathbf{p})\cdot\mathbf{n}_q \,\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
\overset{(7.10)\,\&\,(7.11)}{=}& \frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\theta)\sin^2(\theta)\cos(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{=0}\right)\frac{\partial\phi(\mathbf{p})}{\partial\eta} \\
&+ \frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\theta)\sin^2(\theta)\sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{=0}\right)\frac{\partial\phi(\mathbf{p})}{\partial\xi} \\
&+ \frac{e^{ik\varepsilon}}{4\pi}(1-ik\varepsilon)\left(\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{\frac{2\pi}{3}}\right)\frac{\partial\phi(\mathbf{p})}{\partial n_p} \\
=& \frac{e^{ik\varepsilon}}{6}(1-ik\varepsilon)\frac{\partial\phi(\mathbf{p})}{\partial n_p} \\
\Longrightarrow \quad \lim_{\varepsilon\to 0} H_{2\,2} =& \frac{1}{6}\frac{\partial\phi(\mathbf{p})}{\partial n_p} = \frac{1}{6}\upsilon(\mathbf{p})
\end{aligned}
$$

To regain perspective:

$$
H_1 + H_2 = I_1 + I_2 + \frac{\partial\phi^{in}(\mathbf{p})}{\partial n_p} + \int_{\partial\widetilde{U}\setminus\triangle U_{\mathbf{p}}} \left(\frac{\partial^2 G(\mathbf{p},\mathbf{q})}{\partial n_q \partial n_p}\phi(\mathbf{q}) - \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p}\upsilon(\mathbf{q})\right)\mathrm{d}q
$$

7.2 The combined linear system

In this section we will give the resulting form of the linear system for the improved boundary integral method. The ansatz for transforming this general system into the standard form is very briefly mentioned again and a reference is given. The resulting normal derivative boundary equation (7.1) for $\mathbf{p}_i \in \partial\widetilde{U}$ and constant planar triangles written-out:

$$\frac{1}{2}\upsilon_i + \sum_{\substack{j=1\\j\neq i}}^{n} \left(\int\limits_{\triangle U_j} \frac{\partial G(\mathbf{p}_i,\mathbf{q})}{\partial n_{p_i}}\,\mathrm{d}q \right) \upsilon_j = \frac{\partial \phi^{in}(\mathbf{p}_i)}{\partial n_{p_i}} + \sum_{\substack{j=1\\j\neq i}}^{n} \left(\int\limits_{\triangle U_j} \frac{\partial^2 G(\mathbf{p}_i,\mathbf{q})}{\partial n_q \partial n_{p_i}}\,\mathrm{d}q \right) \phi_j + \left(\frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)}\,\mathrm{d}\theta \right) \phi_i$$

The Burton-Miller combined boundary equation (7.2) for $\mathbf{p}_i \in \partial\widetilde{U}$ and constant planar triangles written-out:

$$\begin{aligned}\left(\frac{i}{2k} + \frac{i}{4\pi k}\int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta + \alpha\frac{1}{2} \right) \upsilon_i + \sum_{\substack{j=1\\j\neq i}}^{n} \int\limits_{\triangle U_j} \left(G(\mathbf{p}_i,\mathbf{q}) + \alpha \frac{\partial G(\mathbf{p}_i,\mathbf{q})}{\partial n_{p_i}}\,\mathrm{d}q \right) \upsilon_j \\ = \phi^{in}(\mathbf{p}_i) + \alpha\frac{\partial \phi^{in}(\mathbf{p}_i)}{\partial n_{p_i}} + \left(-\frac{1}{2} + \alpha \left(\frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)}\,\mathrm{d}\theta \right) \right) \phi_i \\ + \sum_{\substack{j=1\\j\neq i}}^{n} \left(\int\limits_{\triangle U_j} \left(\frac{\partial G(\mathbf{p}_i,\mathbf{q})}{\partial n_q} + \alpha \frac{\partial^2 G(\mathbf{p}_i,\mathbf{q})}{\partial n_q \partial n_{p_i}} \right) \mathrm{d}q \right) \phi_j\end{aligned} \tag{7.15}$$

The resulting Burton-Miller combined boundary equation system:

$$(A + \alpha B)\,\boldsymbol{\upsilon}_{vec} = \left(\boldsymbol{\phi}^{in}_{vec} + \alpha \boldsymbol{\upsilon}^{in}_{vec}\right) + (C + \alpha D)\,\boldsymbol{\phi}_{vec}$$

with A, B, C and $D \in \mathbb{C}^{n\times n}$ and $\boldsymbol{\phi}_{vec}$, $\boldsymbol{\upsilon}_{vec}$, $\boldsymbol{\phi}^{in}_{vec}$ and $\boldsymbol{\upsilon}^{in}_{vec} \in \mathbb{C}^n$.

$$\boldsymbol{\upsilon}_{vec} := \begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_n \end{pmatrix} \quad \boldsymbol{\phi}^{in}_{vec} := \begin{pmatrix} \phi^{in}(\mathbf{p}_1) \\ \vdots \\ \phi^{in}(\mathbf{p}_n) \end{pmatrix} \quad \boldsymbol{\upsilon}^{in}_{vec} := \begin{pmatrix} \frac{\partial \phi^{in}(\mathbf{p}_1)}{\partial n_{p_1}} \\ \vdots \\ \frac{\partial \phi^{in}(\mathbf{p}_n)}{\partial n_{p_n}} \end{pmatrix} \quad \boldsymbol{\phi}_{vec} := \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

$$A := \begin{pmatrix} \frac{i}{2k} + \frac{i}{4\pi k}\int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta_1 & \int\limits_{\triangle U_2} \partial G(\mathbf{p}_1,\mathbf{q})\,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \partial G(\mathbf{p}_1,\mathbf{q})\,\mathrm{d}q & \int\limits_{\triangle U_n} \partial G(\mathbf{p}_1,\mathbf{q})\,\mathrm{d}q \\ \int\limits_{\triangle U_1} \partial G(\mathbf{p}_2,\mathbf{q})\,\mathrm{d}q & \frac{i}{2k} + \frac{i}{4\pi k}\int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta_2 & \ddots & \int\limits_{\triangle U_{n-1}} \partial G(\mathbf{p}_2,\mathbf{q})\,\mathrm{d}q & \int\limits_{\triangle U_n} \partial G(\mathbf{p}_2,\mathbf{q})\,\mathrm{d}q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \int\limits_{\triangle U_1} \partial G(\mathbf{p}_{n-1},\mathbf{q})\,\mathrm{d}q & \int\limits_{\triangle U_2} \partial G(\mathbf{p}_{n-1},\mathbf{q})\,\mathrm{d}q & \ddots & \frac{i}{2k} + \frac{i}{4\pi k}\int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta_{n-1} & \int\limits_{\triangle U_n} \partial G(\mathbf{p}_{n-1},\mathbf{q})\,\mathrm{d}q \\ \int\limits_{\triangle U_1} \partial G(\mathbf{p}_n,\mathbf{q})\,\mathrm{d}q & \int\limits_{\triangle U_2} \partial G(\mathbf{p}_n,\mathbf{q})\,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \partial G(\mathbf{p}_n,\mathbf{q})\,\mathrm{d}q & \frac{i}{2k} + \frac{i}{4\pi k}\int\limits_0^{2\pi} e^{ikR(\theta)}\,\mathrm{d}\theta_n \end{pmatrix}$$

$$B := \begin{pmatrix} \frac{1}{2} & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_{p_1}} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_{p_1}} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_{p_1}} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_{p_2}} \,\mathrm{d}q & \frac{1}{2} & \ddots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_{p_2}} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_{p_2}} \,\mathrm{d}q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_{p_{n-1}}} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_{p_{n-1}}} \,\mathrm{d}q & \ddots & \frac{1}{2} & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_{p_{n-1}}} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_{p_n}} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_{p_n}} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_{p_n}} \,\mathrm{d}q & \frac{1}{2} \end{pmatrix}$$

$$C := \begin{pmatrix} -\frac{1}{2} & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_1,\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & -\frac{1}{2} & \ddots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_2,\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \ddots & -\frac{1}{2} & \int\limits_{\triangle U_n} \frac{\partial G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial G(\mathbf{p}_n,\mathbf{q})}{\partial n_q} \,\mathrm{d}q & -\frac{1}{2} \end{pmatrix}$$

$$D := \begin{pmatrix} \frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \,\mathrm{d}\theta_1 & \int\limits_{\triangle U_2} \frac{\partial^2 G(\mathbf{p}_1,\mathbf{q})}{\partial n_q \partial n_{p_1}} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial^2 G(\mathbf{p}_1,\mathbf{q})}{\partial n_q \partial n_{p_1}} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial^2 G(\mathbf{p}_1,\mathbf{q})}{\partial n_q \partial n_{p_1}} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial^2 G(\mathbf{p}_2,\mathbf{q})}{\partial n_q \partial n_{p_2}} \,\mathrm{d}q & \frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \,\mathrm{d}\theta_2 & \ddots & \int\limits_{\triangle U_{n-1}} \frac{\partial^2 G(\mathbf{p}_2,\mathbf{q})}{\partial n_q \partial n_{p_2}} \,\mathrm{d}q & \int\limits_{\triangle U_n} \frac{\partial^2 G(\mathbf{p}_2,\mathbf{q})}{\partial n_q \partial n_{p_2}} \,\mathrm{d}q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \int\limits_{\triangle U_1} \frac{\partial^2 G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q \partial n_{p_{n-1}}} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial^2 G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q \partial n_{p_{n-1}}} \,\mathrm{d}q & \ddots & \frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \,\mathrm{d}\theta_{n-1} & \int\limits_{\triangle U_n} \frac{\partial^2 G(\mathbf{p}_{n-1},\mathbf{q})}{\partial n_q \partial n_{p_{n-1}}} \,\mathrm{d}q \\ \int\limits_{\triangle U_1} \frac{\partial^2 G(\mathbf{p}_n,\mathbf{q})}{\partial n_q \partial n_{p_n}} \,\mathrm{d}q & \int\limits_{\triangle U_2} \frac{\partial^2 G(\mathbf{p}_n,\mathbf{q})}{\partial n_q \partial n_{p_n}} \,\mathrm{d}q & \cdots & \int\limits_{\triangle U_{n-1}} \frac{\partial^2 G(\mathbf{p}_n,\mathbf{q})}{\partial n_q \partial n_{p_n}} \,\mathrm{d}q & \frac{ik}{2} - \int\limits_0^{2\pi} \frac{e^{ikR(\theta)}}{4\pi R(\theta)} \,\mathrm{d}\theta_n \end{pmatrix}$$

The $\mathrm{d}\theta_i$ on the i-th diagonal entry of the matrices A and D also signifies, that the $R(\theta)$ function on the i-th triangle $\triangle U_i$ is meant.

As mentioned earlier, the general solution strategy for the resulting matrices $(A + \alpha B)$ and $(C + \alpha D)$ is to substitute ϕ_i into υ_i or vice versa through their boundary condition.

$$a_i \phi_i + b_i \upsilon_i = f_i \quad \text{for } i = 1, \ldots, n$$

The above method is outlined in detail in [Kir]. Once the boundary states have been solved for, the solution for a point $\mathbf{p}$ in the interior of the domain can be acquired via:

$$\phi(\mathbf{p}) = \phi^{in}(\mathbf{p}) + \sum_{j=1}^{n} \int\limits_{\triangle U_j} \left(\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q} \phi_j - G(\mathbf{p},\mathbf{q}) \upsilon_j \right) \mathrm{d}q \tag{7.16}$$

Note that $p \notin \partial\widetilde{U}$ for (7.16), therefore all integrals can be solved directly via standard Gaussian quadrature methods for planar triangles.

8 The implementation

The collocation boundary element method was implemented in a program called 3d_BEM. The program consists of a simple script interpreter, a graphic user interface with 3d viewer and the boundary element solver. The program enables a flexible setup of small-scale acoustic simulations. One can define 3d mesh geometries in scripts or register mesh files for complex geometries. Those geometries can then be manipulated and the mesh can be refined to improve accuracy. Once the boundary data has been solved for, observation fields can be solved and visualized. Some simulations with 3d_BEM are presented in the following.

8.1 Radiating puck

The sound field from a puck of diameter 10 meters, with radiating boundary conditions on top and Neumann boundary conditions on the other sides, was simulated at different frequencies. The colour gradient from red (the maximum) to blue spans approximately $20db$ of sound pressure difference for each figure. The sound pressure is indistinctly lower in the dark blue areas. λ states the wavelength. Note the interference pattern on figure 8.2.

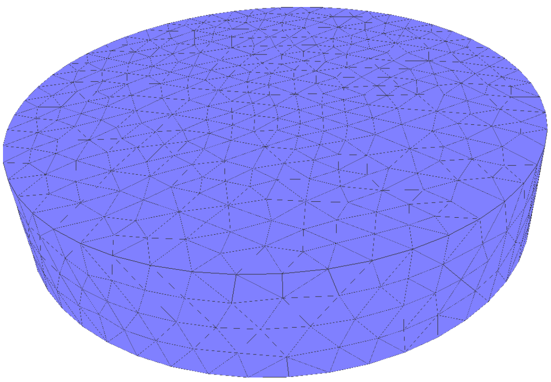

Figure 8.1: The puck

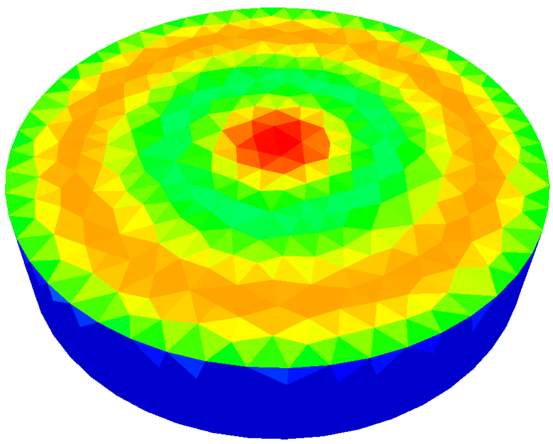

Figure 8.2: 100 hz
$\lambda \approx 3.43m$

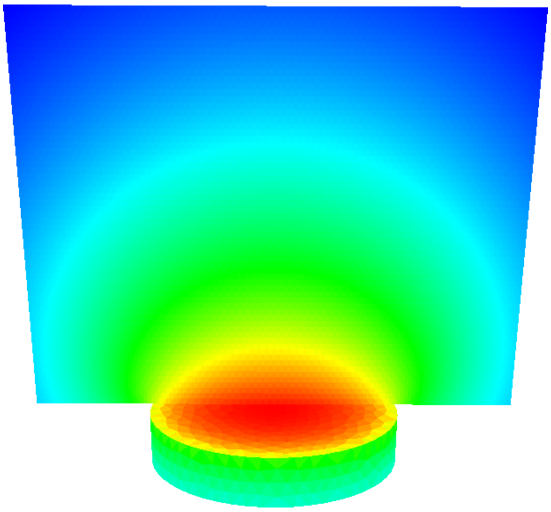

Figure 8.3: 6.25 hz
$\lambda \approx 54.88m$

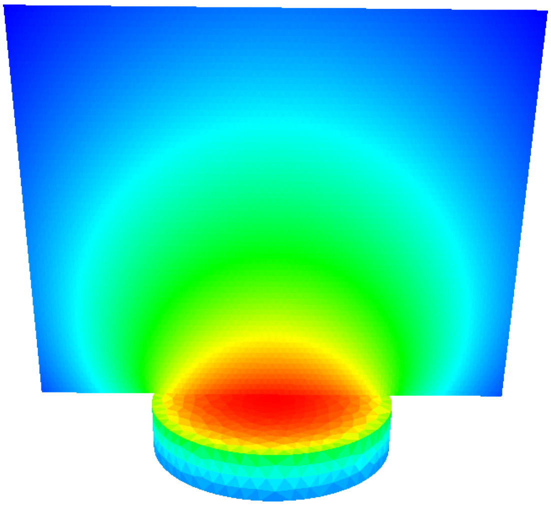

Figure 8.4: 12.5 hz
$\lambda \approx 27.44m$

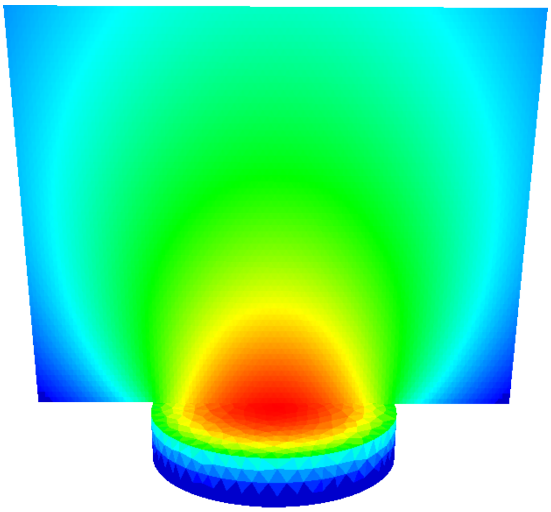

Figure 8.5: 25 hz
$\lambda \approx 13.72m$

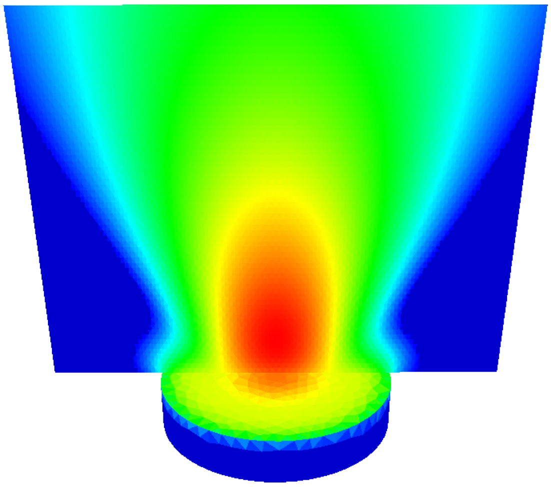

Figure 8.6: 50 hz
$\lambda \approx 6.83m$

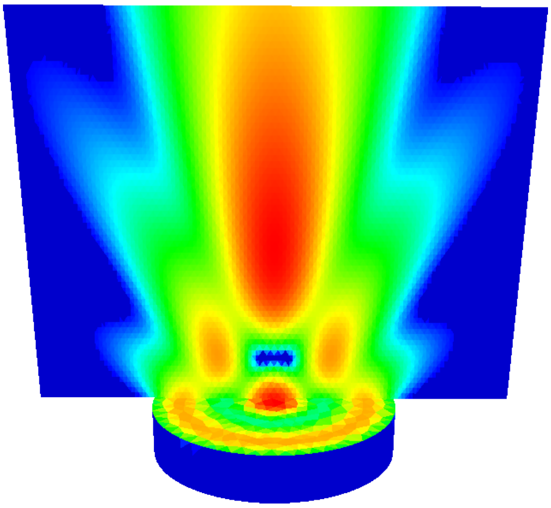

Figure 8.7: 100 hz
$\lambda \approx 3.43m$

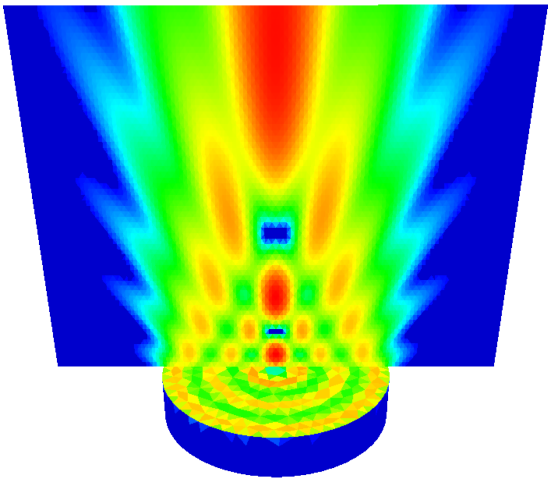

Figure 8.8: 200 hz
$\lambda \approx 1.715m$

The simulation shows the characteristic transition from wide radiation to beaming behaviour for a plane constant-phase source with rising frequency.

8.2 Spherical scatterer

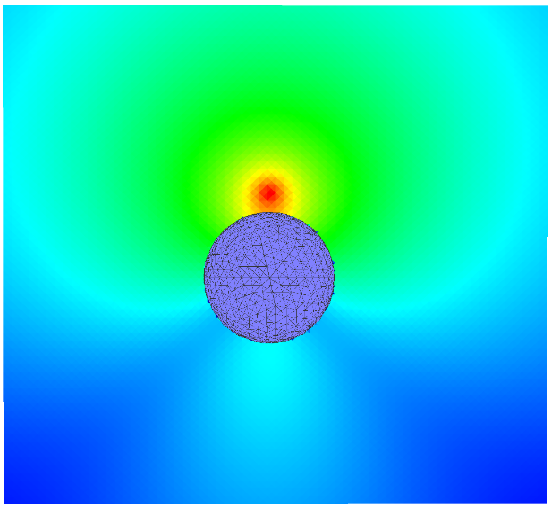

Figure 8.9: 10 hz
$\lambda \approx 34.3m$

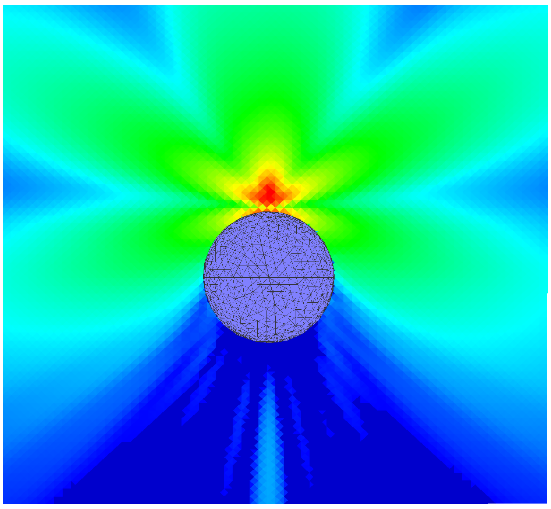

Figure 8.10: 100 hz
$\lambda \approx 3.43m$

The simulation shows a point source field, scattered by a sound hard sphere of 10m diameter. The red-blue gradient range is $40db$.

9 Conclusion

The presented collocation method is only the crudest form of the boundary element method, but due to its relative simplicity and computational efficiency it is widely used in practise. In the following we give some observations on its application, mention important omissions and introduce open questions and further ideas.

9.1 Theory on the Helmholtz problem

Though not covered in this text, but essential to the boundary element method for Helmholtz problems is the mathematical theory for the Helmholtz problem itself. Apart from Sommerfelds radiation condition, we have not examined the question of well-posedness for the exterior and interior Helmholtz problem with the different boundary conditions or source terms. This demands further investigation.

9.2 The coupling parameter

The coupling parameter $\alpha \in \mathbb{C}$ of the Burton-Miller combined boundary equation (7.2) poses an open question. In [Mar15, page 17] the author states, that an the exterior Neumann problem with Sommerfelds radiation condition has a unique solution and that 'an optimal choice of the coupling parameter depends on the frequency and, most likely, on the problem. Therefore, the search for an optimal coupling parameter appears to be very complex ...'
While referring to the same Neumann problem, the author of [CCH09, page 165] states, that 'it can be shown that provided that the imaginary part of α is non-zero, then [the Burton-Miller combined boundary equation] has a unique solution for all real and positive k.' The author of [ZCGD15, page 50] also takes the latter postition and gives further insight into how the combined boundary equation supposedly penalises the fictious eigenfrequencies. This subject has to be further investigated by the author of this text. Furthermore answers to the problem of uniqueness for impedance boundary conditions are sought.

9.3 Convergence of the method

Even though the collocation method is widely used in practice, there has not been a publication of generally valid convergence statements. There exist so far only wider convergence statements for higher order methods (at least linearly approximating test functions on the boundary) [Sto04, page 27]. As a rule of thumb for 'engineering accuracy', the author of [CL07, page 30] gives 5-10 elements per wavelength in each direction.

9.4 Computational observations

The arising matrices of the boundary element method are generally fully populated, asymmetric and of size $n \times n$ with regard to the number of boundary elements. This imposes severe limitations on the magnitude of the computable problems due to time and memory constraints. The duration of straightforwardly solving the resulting linear system via LU-decomposition and forward and backward substitution is in the order of magnitude n^3. Therefore a basic boundary element method is realisticly limited to a few thousand boundary elements on standard computer hardware. An important observation is, that all matrix entries are independent of each other and therefore the problem can be considered *embarrassingly parallel*. There have been multiple very successful advances in compressing the information in large boundary element matrices, one notable being the *fast multipole method*. In the *fast multipole method*, elements that lie close together are grouped to a single source with regard to their influence on other elements that are further away. The resulting complexity is given in [Sto04, page 71] as $O(n \log_2(n))$. The authors of the open-source BEM Matlab/C++ toolbox, called NiHu, claim that their *fast multipole method* accelerated toolbox 'can solve problems of industrial size on a single desktop computer' [FR14, page 10] and demonstrate an example problem with 200.000 elements.

9.5 Further ideas

This paper concludes with a brief mention of two interesting ideas. Generally the domain U can be decomposed in smaller subdomains, which are then interfaced via boundary conditions on the 'artificial boundaries'. These boundary conditions impose equality constraints on the sound pressure and the particle velocity on both sides. An approach is outlined in [SW11]. Also mentionable is that there exist also Green's functions for the wave equation, which enables the derivation of direct time-domain boundary element methods. This is in contrast to the earlier mentioned ansatz to reconstruct the time-domain behaviour of the solution by Fourier-synthesis with frequency-domain (Helmholtz) solutions.

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